Problem Set

Introduction to Cosmology

Spring, 2018

1. Prove that the following formulations of the equivalence principle are equivalent:

   "Gravitational mass and inertial mass are equal."

   "Locally gravity is equivalent to a uniformly accelerated motion of inertial frame."

2. A clock is traveling around the Earth on a circular orbit. Another clock remains at rest on the ground. Both clocks tick at the same pace. Find the orbit radius.

3. It is estimated that the planet Earth had formed about $4.5 \cdot 10^9$ yrs ago, so since then there were no significant mixing between the matter at the centre and on the surface. Estimate by how much the matter at the Earth centre is *younger* than the matter on the surface.

4. Prove that a law of motion for a uniformly accelerating observer is

   $$x(t) = \frac{c^2}{a}\sqrt{1 + \left(\frac{at}{c^2}\right)^2},$$

   where c is the speed of light and a is her acceleration in a co-moving frame.

5. Show that components of Riemann tensor inside the Earth are:

   $$R^i_{0j0} = \frac{4\pi G\rho}{3c^2}\delta^i_j,$$

   where ρ is the local mass density of Earth material.

6. A bullet is fired in the RW universe at a speed v_1. Later when the universe has expanded by a factor of $(1 + z)$, the bullet speed becomes v_2. Find the relation between v_1, v_2, and z.

7. Consider two particles of dark matter being initially at rest in empty space far away from any matter, ordinary and dark. Using the accepted value of Hubble constant $H_0 = 68$ km/(s·Mpc) estimate the critical distance between the particles at which they will move apart. Assume that dark matter particles interact only gravitationally and the particle mass is 100 GeV/c^2. In other words, find the distance at which cosmic expansion wins over Newtonian gravitational attraction.

8. Suppose the universe contains only non-relativistic matter. Prove that if the matter density exceeds $\rho_0^{crit} = 3H_0^2/(8\pi G)$, where H_0 is the Hubble constant at present, the universe eventually collapses, if it less or equal, the universe will expand indefinitely.

9. Assume that $\Omega_K = \Omega_R = 0$, i.e. $\Omega_\Lambda + \Omega_M = 1$. Show that for $z \ll 1$ the luminosity distance is:

$$d_L \approx \frac{c}{H_0}[z + (\Omega_\Lambda + \frac{1}{4}\Omega_M)z^2 + o(z^3)].$$

10. Using the Friedmann equations show that *decelaration parameter* q_0 in terms of Ω's is

$$q_0 \equiv -\frac{\ddot{a}_0 a_0}{\dot{a}_0^2} = \frac{1}{2}(\Omega_M - 2\Omega_\Lambda + 2\Omega_R).$$

Find its benchmark value.

11. Assume $\Omega_M = 0.315$ and $\Omega_\Lambda = 0.685$ with Ω_R negligible. What is the redshift at which the expansion of the universe stopped decelerating and began to accelerate? How many years ago did this happen? $H_0^{-1} \approx 6.58 \times 10^9$yrs.

12. Estimate the time passed from the BB to radiation decoupling ($z_L = 1100$) in the Benchmark Model. *Hint:* verify that Ω_Λ and Ω_R can be neglected, so the integral depends only on Ω_M.

13. Estimate the mass of iron core at which neutronization is possible. Use $Q = (m_n - m_p)c^2 \approx 1.3\,\text{MeV}$.

14. According to quantum chromodynamics the physical vacuum is filled with a quark-gluon condensate with energy density $B \approx -200\text{MeV}\cdot\text{fm}^{-3}$ (the negative sign indicates bound state). Estimate the temperature (in MeV) of the phase transition from vacuum to quark-gluon plasma using analogy with boiling water and taking into account only gluons. Consider the condensate as a liquid of zero entropy. A gluon has two polarizations and 8 color states. $\hbar c \approx 200\text{MeV}\cdot\text{fm}$.

15. Estimate the minimal energy of a moving radially inward shell of photons of wavelength λ which is bound to collapse.

16. Using thermodynamic relations, $dE = TdS$ and $E = Mc^2$ obtain the Hawking-Bekenstein formula for the BH entropy:

$$S = \frac{k_B}{4} \frac{4\pi r_s^2}{l_{Pl}^2},$$

where $l_{Pl} \approx 1.6 \times 10^{-33}$cm is the Planck length.