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СЕМЕЙСТВА МНОЖЕСТВ С ЗАПРЕЩЕННЫМИ КОНФИГУРАЦИЯМИ И ПРИЛОЖЕНИЯ К ДИСКРЕТНОЙ ГЕОМЕТРИИ

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Andrey Borisovich Kupavskii

FAMILIES OF SETS WITH FORBIDDEN CONFIGURATIONS AND APPLICATIONS TO DISCRETE GEOMETRY

05.13.17 — theoretical foundations of computer science

Abstract of a dissertation for the degree
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Actuality.

Extremal set theory is a field started by Erdős, Kleitman, Sperner and others, which typically searches for the largest systems of sets (*families*) under certain restrictions. A *family* is simply a collection of sets. One may interpret these question as questions about Boolean functions, and it should come as no surprise that results and notions from extremal set theory have applications in computer science, such as measuring the complexity of an arrangement of objects, the algorithms for efficient encoding and error-free decoding of information, hardness of approximation or building hash functions.

Let me present several key results from the area that motivated my research. Recall that $[n]$ stands for the set of first n integers, $2^{[n]}$ for its power set. One of the first results in the field belongs to Sperner and is as follows.

Theorem (Sperner[1]). *The largest size of a family $\mathcal{F} \subset 2^{[n]}$ such that no $F_1 \in \mathcal{F}$ is strictly contained in $F_2 \in \mathcal{F}$, is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

One of its applications is to the famous Hardy–Littlewood problem on the maximum number of distinct sums of numbers $a_1, \ldots, a_n > 1$ that all fall into some interval of length 1. An applications of completely different flavour is to the design of hash functions for a CPU. The studies of families with forbidden subposets became an active area of research with many challenging open questions.

Another cornerstone of extremal set theory is the following theorem of Erdős, Ko and Rado, proved in 1938, but published only 23 years later, in 1961. We say that a family is *intersecting* if any two of its sets intersect.

Theorem (Erdős, Ko, Rado[2]). *Any intersecting family $\mathcal{F}$ of k-element subsets of $[n]$ has size at most $\binom{n-1}{k-1}$, provided $n \geqslant 2k$.*

The theorem generated a wealth of research. In particular, several different proofs of this theorem were found, which stimulated the development of different areas of combinatorics. Its generalizations, such as the famous Frankl–Wilson theorem and the Complete t-Intersection Theorem due to Ahlswede and Khachatrian[3], have several applications elsewhere in discrete mathematics and computer science: explicit construction of Ramsey graphs (which, in itself, plays an important role in understanding the hardness of the problem of finding the chromatic number of a graph), problems on geometric graphs and geometric Ramsey

[1] E. Sperner, *Ein Satz uber Untermengen einer endlichen Menge,* Math. Z. 27 (1928), N1, 544–548.

[2] P. Erdős, C. Ko, and R. Rado, *Intersection theorems for systems of finite sets*, The Quarterly Journal of Mathematics 12 (1961) N1, 313–320.

[3] R. Ahlswede and L. Khachatrian, *The complete intersection theorem for systems of finite sets*, Eur. J. Comb. 18 (1997), N2, 125–136.

theory, hardness of approximation, which is one of the most exciting and fast growing areas of theoretical computer science for the moment (the celebrated result of Dinur and Safra on hardness of approximation of vertex cover uses the Ahlswede–Khachatrian theorem).

The following result plays a central role in many applications of extremal set theory to different threshold phenomena in combinatorics and computer science. For a family $\mathcal{F}$ of k-element sets, let $\partial\mathcal{F}$ stand for the collection of all $(k-1)$-element sets contained in at least one set from $\mathcal{F}$. Recall that $\binom{x}{k} := \frac{x(x-1)\dots(x-k+1)}{k!}$ for any real $x \geqslant k$. The Kruskal–Katona theorem, in the form found by Lovász, states that

Theorem (Kruskal[4], Katona[5]). *Given a family $\mathcal{F}$ of k-element sets with $|\mathcal{F}| = \binom{x}{k}$ for some $x \geqslant k$, we have $|\partial\mathcal{F}| \geqslant \binom{x}{k-1}$.*

In particular, it was used by Bollobás and Thomason to show that monotone properties always have a threshold. It is also directly related to the famous Kahn–Kalai–Linial theorem[6] from the analysis of Boolean functions.

Let us conclude this section with the Vapnik–Chervonenkis–Sauer–Shelah lemma, which is important in the areas of statistical learning, computational geometry and high-dimensional data analysis. For a family $\mathcal{F} \subset [n]$, its *VC-dimension* is the size of the largest subset $X \subset [n]$, such that X is *shattered*, that is, $\{F \cap X : F \in \mathcal{F}\} = 2^X$. (In plain words, X is shattered if any subset of X, including the empty set and X itself, can be obtained by intersecting X with some member of $\mathcal{F}$.) Many families that arise in geometrical or statistical applications, have bounded VC-dimension. If this is the case, then the size of a family is polynomial in n:

Theorem (Vapnik, Chervonenkis[7]; Sauer; Shelah). *If $\mathcal{F} \subset [n]$ has VC-dimension at most d, then $|\mathcal{F}| \leqslant \sum_{i=0}^{d} \binom{n}{i} = O(n^d)$.*

Since its introduction, VC-dimension became the central concept in learning and related fields. The lemma above, in particular, is crucial for proving the existence of small ϵ-nets and ϵ-approximations, which are provably small-sized samples of the structures that provide a provably good approximation of that structure.

[4] J.B. Kruskal, *The Number of Simplices in a Complex*, Mathematical optimization techniques 251 (1963), 251–278.

[5] G.O.H. Katona, *A theorem of finite sets*, Theory of Graphs, Proc. Coll. Tihany (1966), Akad, Kiado, Budapest (1968); Classic Papers in Combinatorics (1987), 381–401.

[6] J. Kahn, G. Kalai, and N. Linial, *The influence of variables on Boolean functions*, Proc. 29th Annual Symposium on Foundations of Computer Science (FOCS) (1988), 68–80.

[7] V. N. Vapnik and A. Ya. Chervonenkis, *On the uniform convergence of relative frequencies of events to their probabilities*, Theory of Probability and its Applications 16 (1971), N2, 264–280.

Development of the subject

Families with forbidden intersection patterns. The Erdős–Ko–Rado theorem, stated above, is tight on a *full star*: the collection of all k-element subsets of $[n]$ containing a given element. Any subfamily of a full star is a *star*. Erdős, Ko and Rado asked, what is the next largest maximal intersecting family. This question was answered by Hilton and Milner[8], but more general questions lead to a development of a whole field of research. One perspective on this field is as follows. The goal is to determine, how "stable" is the full star: in a given "metric", how large is the largest intersecting family $\mathcal{F}(d) \subset \binom{[n]}{k}$, that is at distance at least d from a star in that "metric". Or, seen from a different perspective, we fix a certain parameter of a family and ask for the largest intersecting family with this parameter falling in a certain range.

Frankl[9] determined the size of the largest intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ in terms of its *maximum degree* $\Delta(\mathcal{F})$: the maximum over the elements of the ground set of the number of sets from $\mathcal{F}$ containing it. Together with Zakharov[10], we got a stronger, "dual" version of Frankl's result in terms of *diversity* $\gamma(\mathcal{F})$, where $\gamma(\mathcal{F}) = |\mathcal{F}| - \Delta(\mathcal{F})$. Similar, but weaker, stability results were obtained by Keevash and Mubayi, Keevash, Das and Tran, Keevash and Long.

Another approach was proposed in the works of Friedgut and Dinur and Friedgut[11] and recently greatly developed in the works of Keller, Lifshitz and coauthors. There, using the machinery of discrete Fourier transform, they proved the following type of results: any intersecting family, up to a small remainder, is contained in an intersecting junta. Here, a j-junta $\mathcal{J}$ is a family such that whether $F \in \mathcal{J}$ or not depends on an intersection of F with a fixed set of size at most j. In these terms, a full star (as well as its complement) is a 1-junta.

Another possible "distance" is the *covering number* $\tau(\mathcal{F})$ of $\mathcal{F}$, that is, the size of the smallest set that intersects all sets in $\mathcal{F}$. One may summarize it in the following problem: find the largest intersecting family in $\binom{[n]}{k}$ with $\tau(\mathcal{F}) \geqslant t$. For $t = 2$, the answer is given by the same result of Hilton and Milner, for $t = 3, 4$ and $n > n_0(k)$ it was solved by Frankl and Frankl, Ota and Tokushige, respectively. In a recent paper, I resolved the question for $t = 3$ and $n > Ck$.

[8] A.J.W. Hilton and E.C. Milner, *Some intersection theorems for systems of finite sets*, Quart. J. Math. Oxford 18 (1967), 369–384.

[9] P. Frankl, *Erdős–Ko–Rado theorem with conditions on the maximal degree*, J. Comb. Theory Ser. A 46 (1987), N2, 252–263.

[10] A. Kupavskii and D. Zakharov, *Regular bipartite graphs and intersecting families*, J. Comb. Theory Ser. A 155 (2018), 180–189.

[11] I. Dinur and E. Friedgut, *Intersecting families are essentially contained in juntas*, Comb. Probab. Comput. 18 (2009), 107–122.

With some work, the same could be done for the $t = 4$ case. However, the case $t \geqslant 5$ remains wide open even for large n.

The particular case $t = k$ of this problem was raised and studied in a classic paper of Erdős and Lovász[12]. Since each set from an intersecting family $\mathcal{F}$ is a cover for the family, we have $\tau(\mathcal{F}) \leqslant k$ for any intersecting family. Erdős and Lovász showed that the largest size of such a family is at most k^k (interestingly, the bound does not depend on the size of the ground set). Recently, this bound was improved by Frankl, but still has the form $k^{(1+o(1))k}$. The lower bound, due to Frankl, Ota and Tokushige, is roughly $(k/2)^k$.

In the study of intersecting families, it is often very helpful to have some analogous results for several families. We say that families $\mathcal{A}, \mathcal{B}$ are *cross-intersecting*, if for any $A \in \mathcal{A}, B \in \mathcal{B}$, A and B intersect. The study of cross-intersecting families goes back to the aforementioned work of Hilton and Milner. There are two types of questions that were studied quite extensively: sum-type inequalities, in which the goal is to bound $|\mathcal{A}| + |\mathcal{B}|$, given some restrictions on one of the families, and product-type inequalities where the goal is to bound $|\mathcal{A}||\mathcal{B}|$. The former were studied by Borg, Frankl and Kupavskii, Frankl and Tokushige, Hilton and Milner, Kupavskii and Zakharov. The latter were studied by Frankl and Kupavskii, Matsumoto and Tokushige, Pyber.

A more general question, addressed by Erdős, Ko, and Rado, is to find the largest t-intersecting family. We say that a family is *t-intersecting* if any two sets from the family intersect in at least t elements. In their paper, they have shown that the size of a t-intersecting family in $\binom{[n]}{k}$ is at most $\binom{n-t}{k-t}$, provided $n > n_0(k, t)$. Later, Frankl formulated a conjecture for all n and k, which, after partial results by Frankl, Wilson, Füredi and Frankl, was completely settled by Ahlswede and Khachatrian. As we have mentioned, this theorem found an application to the problem of hardness of approximation of vertex cover in the work of Dinur and Safra. Of course, it is natural to ask for the cross-intersecting analogues of this result (studied by Frankl and Kupavskii, Frankl, Lee, Siggers and Tokushige, Wang and Zhang), as well as for stability results (studied by Ellis, Keller and Lifshitz and by Friedgut).

Erdős and Sós in 1971 asked the following question: how large can $\mathcal{F} \subset \binom{[n]}{k}$ be, provided no two sets in $\mathcal{F}$ intersect in *exactly* t elements? Recently, a big progress on this question was made by Ellis, Keller and Lifshitz, who showed that for any fixed t and a very wide range of $k = k(n)$, the answer to this question is

[12] P. Erdős and L. Lovász, *Problems and results on 3-chromatic hypergraphs and some related questions*, Infinite and finite sets 10 (1975), N2, 609–627.

the same as in the theorem of Ahlswede and Khachatrian. In this dissertation, however, I am particularly interested in the case when t is comparable to n. In this case, there is a beautiful result of Frankl and Wilson[13], which shows that for a certain choice of parameters, the size of such family is *exponentially in* n smaller than $\binom{[n]}{k}$. Interestingly, this theorem require certain numbers to be prime powers. Without that restriction, the bounds in the theorem are false. However, there is a qualitatively similar result of Frankl and Rödl for a pair of families that, albeit gives worse quantitative bounds, is much more flexible and, in particular, does not have any algebraic restrictions. Recently, Keevash and Long managed to deduce the result of Frankl and Rödl from the result of Frankl and Wilson using the modern probabilistic technique of *dependent random choice*.

The result of Frankl and Wilson allows to find explicit constructions for Ramsey numbers. In combinatorial geometry, it is applied to the problems of finding the chromatic number of Euclidean space and Borsuk's problem. The chromatic number of $\mathbb{R}^d$ is the minimum number of colors needed to color all points of $\mathbb{R}^d$ so that no two points at unit distance apart receive the same color. This quantity was studied quite extensively (see the surveys of Raigorodskii).

Raigorodskii[14] succeeded in improving the bounds in the aforementioned geometric problems by enlarging the scope of vectors from $\{0,1\}$-vectors to $\{0,\pm1\}$-vectors and proving Frankl–Wilson type theorems for such vectors (see also the works of Raigorodskii with Cherkashin, Kupavskii, Mitricheva, and Ponomarenko). Motivated by this, in a series of works I and Frankl initiated systematic studies of intersecting theorems for families of $\{0,\pm1\}$-vectors. We note here that intersecting theorems were studied for other objects, such as subspaces and permutations, but this is beyond the scope of this dissertation.

Let us return to the first question we have discussed in this section: stability of the Erdős–Ko–Rado theorem. Another way of looking at stability is by studying the so-called "transference" results in Kneser graphs. Let us first give some definitions. A Kneser graph $KG_{n,k}$ is a graph with vertex set $\binom{[n]}{k}$ and edge set formed by all pairs of disjoint sets from $\binom{[n]}{k}$. Note that the Erdős–Ko–Rado theorem determines the size of the largest independent set in $KG_{n,k}$. Similarly to the classical random graph model $G(n,p)$ (cf., e.g., the books of Alon and Spencer and Bollobás), define the *random Kneser graph* $\mathcal{K}G_{n,k}(p)$ as follows:

[13] P. Frankl and R.M. Wilson, *Intersection theorems with geometric consequences*, Combinatorica 1 (1981), N4, 357–368.

[14] A.M. Raigorodskii, *The Borsuk problem and the chromatic numbers of some metric spaces*, Russian Math. Surveys 56 (2001), N1, 103–139.

$V(KG_{n,k}(p)) = V(KG_{n,k})$ and the set of edges of $KG_{n,k}(p)$ is a subset of the set of edges of $KG_{n,k}$, obtained by including each edge from $KG_{n,k}$ independently and with probability p.

Returning to transference, in general, we speak of transference if a certain combinatorial result (with high probability) holds with no changes in the random setting. Studying this phenomenon in the context of the independence number and the chromatic number of generalized Kneser graphs was suggested by Bogolyubskiy, Gusev, Pyaderkin and Raigorodskii[15]. One example of such theorem is due to B. Bollobás, B. Narayanan and A. Raigorodskii[16]. They studied the size of maximal independent sets in $KG_{n,k}(p)$, and showed that for a wide range of parameters the independence number of $KG_{n,k}(p)$ is *exactly* the same as that of $KG_{n,k}$: $\binom{n-1}{k-1}$. Later on, their result was further strengthened by Balogh, Bollobásh and Narayanan, Das and Tran, and by Delvin and Kahn.

Families with forbidden configurations. We say that a family $\mathcal{F}$ *has matching number* s if s is the maximum number of pairwise disjoint sets in $\mathcal{F}$. Intersecting families have matching number 1, and for the edge sets of graphs this definition corresponds to the standard notion of a matching. Erdős and Kleitman in the 60's suggested the following two problems that generalize the Erdős–Ko–Rado theorem. The first one is as follows.

Problem (Erdős, Kleitman[17]). *Determine the size of the largest family* $\mathcal{F} \subset 2^{[n]}$ *with matching number* s.

Kleitman resolved the problem above for $n = sm - 1, sm$, where m is an integer. Queen resolved the only remaining case for $s = 3$: $n = 3m + 1$. For a while, there was no progress. In a recent work with Frankl, we managed to resolve this problem in many new cases.

The more famous question on matchings deals with uniform families.

Problem (Erdős Matching Conjecture[18]). *The size* $e_k(n, s)$ *of the largest family* $\mathcal{F} \subset \binom{[n]}{k}$ *with matching number* s *is* $\max\left\{\binom{n}{k} - \binom{n-s}{k}, \binom{sk+k-1}{k}\right\}$.

The Erdős Matching Conjecture, or EMC for short, is trivial for $k = 1$ and was proved by Erdős and Gallai for the graph case $k = 2$ (this result has stimulated

[15] L.I. Bogolyubskiy, A.S. Gusev, M.M. Pyaderkin, and A.M. Raigorodskii, *Independence numbers and chromatic numbers of random subgraphs in some sequences of graphs*, Doklady Math 90 (2014), N1, 462–465.

[16] B. Bollobás, B.P. Narayanan, and A.M. Raigorodskii, *On the stability of the Erdős–Ko–Rado theorem*, J. Comb. Theory Ser. A 137 (2016), 64–78.

[17] D.J. Kleitman, *Maximal number of subsets of a finite set no k of which are pairwise disjoint*, J. Comb. Theory 5 (1968), 157–163.

[18] P. Erdős, *A problem on independent r-tuples*, Ann. Univ. Sci. Budapest. 8 (1965), 93–95.

a series of papers in graph theory, cf. e.g., the recent paper of Luo). After a series of works, it was completely resolved for $k = 3$ by Frankl. The case $s = 1$ is the Erdős-Ko-Rado theorem.

In his original paper, Erdős proved the conjecture for $n \geqslant n_0(k, s)$. His result was sharpened by Bollobás, Daykin and Erdős, who verified it for $n \geqslant 2k^3 s$. Subsequently, Hao, Loh and Sudakov proved the EMC for $n \geqslant 3k^2 s$. Their proof relies in part on the "multipartite version" of the following universal bound from Frankl's survey on shifting[19]:

$$e_k(n, s) \leqslant s\binom{n-1}{k-1}.$$

If $n = k(s + 1)$ then the right hand side of the equation above is equal to $|\mathcal{A}_0(s, k)|$. For this case, the EMC was implicitly proved by Kleitman in his studies of the non-uniform problem. This was extended very recently by Frankl, who showed that $e_k(n, s) \leqslant \binom{k(s+1)-1}{k}$ for all $n \leqslant (s+1)(k+\epsilon)$, where ϵ depends on k. Improving the aforementioned bounds, Frankl[20] resolved the conjecture for $n \geqslant (2s + 1)k - s$. In a recent work with Frankl[21], we managed to get the current best bound for the EMC: we showed that it is valid for any $s \geqslant s_0$ and $n > \frac{5}{3}sk - \frac{2}{3}s$.

The Erdős Matching Conjecture takes a central place in extremal combinatorics and attracted a lot of attention recently, in particular, due to connections with Dirac thresholds, theory on deviations of sums of random variables and some computer science problems. We will just briefly overview the questions on Dirac thresholds. This active area of research in extremal combinatorics stems from the famous Dirac's criterion for Hamiltonicity: any n-vertex graph with minimum degree at least $n/2$ contains a Hamilton cycle. The generic question is as follows: what is the condition on the minimum degree[22] for a family in order for it to contain a certain spanning (that is, covering the ground set) structure. Probably the most basic structure is a perfect matching, and it turns out that the results for the EMC imply the results on Dirac thresholds for perfect matchings. For details on this connection, we refer to the paper of Alon et. al. and my paper with Frankl, and for more on Dirac thresholds we refer to a recent survey of Zhao[23].

[19] P. Frankl, *The shifting technique in extremal set theory*, Surveys in combinatorics, Lond. Math. Soc. Lecture Note Ser. 123 (1987), 81–110, Cambridge University Press, Cambridge.

[20] P. Frankl, *Proof of the Erdős matching conjecture in a new range*, Isr. J. Math. 222 (2017), N1, 421–430.

[21] P. Frankl and A. Kupavskii, *The Erdős Matching Conjecture and Concentration inequalities*, arXiv:1806.08855

[22] We note here that it may be the degree of an element, or a collective degree of a d-tuple of elements, that is, the number of sets from the family containing that d-tuple

[23] Y. Zhao, *Recent advances on dirac-type problems for hypergraphs*, In Recent Trends in Combinatorics,

One may think of intersecting families or families with matching number s as of families with forbidden configurations: in the first case, the forbidden configuration is 2 disjoint sets, and in the second case it is $s+1$ pairwise disjoint sets. One may similarly interpret the other restrictions on intersections. Having this perspective, it is natural to ask, what could we say about other potential forbidden configurations.

The family $\mathcal{F} \subset 2^{[n]}$ is called *partition-free* if there are no $A, B, C \in \mathcal{F}$ satisfying $A \cap B = \emptyset$ and $C = A \cup B$. Half a century ago Kleitman proved the following beautiful result.

Theorem (Kleitman[24]). *Suppose that $n = 3m+1$ for some positive integer m. Let $\mathcal{F} \subset 2^{[n]}$ be partition-free. Then*

$$|\mathcal{F}| \leqslant \sum_{t=m+1}^{2m+1} \binom{n}{t}.$$

He asked, what is the answer for $n = 3m, 3m+2$. In a recent work with Frankl[25] we used the methods developed for the first problem from this section and completely resolved this problem.

Kleitman considered the following related problem. What is the maximum size $u(n)$ of a family $\mathcal{F} \subset 2^{[n]}$ without three distinct members satisfying $A \cup B = C$. The difference with partition-free families is that one does not require A and B to be disjoint. Kleitman proved $u(n) \leqslant \binom{n}{\lfloor n/2 \rfloor}(1 + \frac{c}{n})$ for some absolute constant c.

An "abstract" version of this problem was solved by Katona and Tarjan. Let $v(n)$ denote the maximum size of a family $\mathcal{F}$ without three distinct members A, B, C such that $A \subset C$ and $B \subset C$. Katona and Tarjan proved that $v(2m+1) = 2\binom{2m}{m}$.

This result was the starting point of a lot of research. The central problem might be stated as to determine the largest size of subsets of the boolean lattice without a certain subposet. We refer the reader to the survey of Griggs and Lee[26]. One of the important recent advancements in the topic was the result of Methuku and Pálvölgyi[27], who showed that for any finite poset there exists a

volume 159 of the IMA Volumes in Mathematics and its Applications. Springer, New York, 2016.

[24] D.J. Kleitman, *On families of subsets of a finite set containing no two disjoint sets and their union*, J. Combin. Theory 5 (1968), N3, 235–237.

[25] P. Frankl and A. Kupavskii, *Partition-free families of sets*, accepted at Proc. London Math. Soc., arXiv:1706.00215

[26] J.R. Griggs, W.T. Li, *Progress on poset-free families of subsets*, Recent Trends in Combinatorics, Springer International Publishing (2016), 317–338.

[27] A. Methuku, D. Pálvölgyi, *Forbidden hypermatrices imply general bounds on induced forbidden subposet problems*, Combinatorics, Probability and Computing 26 (2017), N4, 593–602.

constant C, such that the largest size of a family without an induced copy of this poset has size at most $C\binom{n}{\lfloor n/2\rfloor}$. However, the value of C is unknown in most cases, including the *diamond* poset: four sets A, B, C, D, where $A \subset B \subset D$, $A \subset C \subset D$.

ϵ-nets. Extremal set theory has important applications in computational geometry and statistical learning, thanks to the notions of ϵ-nets and VC-dimension.

Let X be a finite set and let $\mathcal{F}$ be a system of subsets of an X. The pair $(X, \mathcal{F})$ is usually called a *range space*. A subset $X' \subseteq X$ is called an *ϵ-net* for $(X, \mathcal{F})$ if $X' \cap F \neq \emptyset$ for every $F \in \mathcal{F}$ with at least $\epsilon|X|$ elements. (In words, ϵ-net is a hitting set for all large sets from the range space.) The use of small-sized ϵ-nets in geometrically defined range spaces has become a standard technique in discrete and computational geometry, with many combinatorial and algorithmic consequences. In most applications, ϵ-nets precisely and provably capture the most important quantitative and qualitative properties that one would expect from a random sample. Typical applications include the existence of spanning trees and simplicial partitions with low crossing number, upper bounds for discrepancy of set systems, LP rounding, range searching, streaming algorithms; see the books of Matoušek and of Pach and Agarval.

A remarkable result due to Haussler and Welzl[28], based on a ground-breaking work of Vapnik and Chervonenkis, states that if $VC(\mathcal{F}) \leqslant d$ then the minimum size of an ϵ-net is $O(\frac{d}{\epsilon}\log\frac{1}{\epsilon})$. It was shown by Komlós, Pach, and Woeginger that this bound is essentially tight for random range spaces. Over the past two decades, a number of specialized techniques have been developed to show the existence of small-sized ϵ-nets for geometrically defined range spaces (see the works of Aronov, Clarkson, Ezra, Matoušek, Mustafa, Sharir, Varadarajan and others). Based on these successes, it was generally believed that in most geometric scenarios one should be able to substantially strengthen the ϵ-net theorem, and obtain perhaps even a $O\left(\frac{1}{\epsilon}\right)$ upper bound for the size of the smallest ϵ-nets. Similar questions were raised in the statistical learning community. However, in 2012 Alon[29] gave a slightly superlinear lower bound for the size of ϵ-nets for range spaces induced by points and lines In a recent breakthrough, Balogh and Solymosi applied the recently developed container method (coming from extremal combinatorics) to give the lower bound of $\Omega(\frac{1}{\epsilon}\log^{1/3-o(1)}\frac{1}{\epsilon})$ for the smallest size of an ϵ-net for range spaces induced by lines and points.

[28] D. Haussler and E. Welzl, *ϵ-nets and simplex range queries*, Disc. Comput. Geom. 2 (1987), 127–151.

[29] N. Alon, *A Non-linear Lower Bound for Planar Epsilon-nets,* Disc. Comput. Geom. 47 (2012), N2, 235–244.

Shortly after Alon's paper, Pach and Tardos[30] constructed a range space induced by points and halfspaces in $\mathbb{R}^4$ with the smallest ϵ-net of (the worst possible) size $\Omega(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$.

On the other hand, following the work of Clarkson and Varadarajan, it has been gradually realized that if one replaces the condition that the range space $(X, \mathcal{R})$ has bounded VC-dimension by a more refined combinatorial property, one can prove the existence of ϵ-nets of size $o(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$. This lead to the introduction of the new complexity measure: *shallow cell complexity*. A series of elegant results of Aronov, Ezra and Sharir, Chan, Grant, Könemann and Ray, Varadarajan[31], and Mustafa, Dutta and Ghosh[32] illustrate that if the shallow-cell complexity of a set system is $\varphi(n) = o(n)$, then it permits smaller ϵ-nets than what is guaranteed by the ϵ-net theorem.

The main goals of the research

The main goal of my research was to better understand the structure of extremal families with forbidden configurations and to apply these findings to different problems in discrete and computational geometry. To do so, I addressed the following questions.

1) How stable is the Erdős–Ko-Rado theorem? More precisely, how large could an intersecting family be in terms of its distance from the closest star?
2) How far could an intersecting family be from a star?
3) What is the structure of a typical intersecting family?
4) How big could a pair of cross-intersecting or cross s-intersecting families be?
5) Are there analogues of the Erdős–Ko–Rado and Ahlswede–Khachatrian theorems for families of $\{0, \pm 1\}$-vectors?
6) What is the largest size of a family with no matching of size s? What kind of structure does it have?

[30] J. Pach and G. Tardos, *Tight lower bounds for the size of epsilon-nets,* J. Amer. Math. Soc. 26 (2013), N3, 645–658.
[31] K. Varadarajan, *Weighted geometric set cover via quasi uniform sampling*, In Proceedings of the Symposium on Theory of Computing (STOC) (2010), 641–648, New York, USA. ACM.
[32] N. H. Mustafa, K. Dutta, and A. Ghosh, *A simple proof of optimal epsilon-nets*, Combinatorica 38 (2018), N5, 1269–1277.

7) What are the analogues of the Hilton–Milner theorem for the Erdős Matching Conjecture?

8) How does the chromatic number of random Kneser graphs and hypergraphs behave?

9) What is the size of the smallest ϵ-net for range spaces, defined by halfspaces in $\mathbb{R}^d$?

10) How big could the smallest ϵ-net be, given a bound on the shallow cell complexity of a range space?

11) Are there geometric obstructions to finding counterexamples to Borsuk's conjecture?

12) Are there distance graphs with large chromatic number and with no large cliques or large girth?

The main results of the dissertation

1) I have found an essentially sharp upper bound on the size of an intersecting family in terms of its distance from the closest star (diversity).

2) I have shown that the diversity of an intersecting family is at most $\binom{n-3}{k-2}$, provided $n > Ck$.

3) I have showed that a typical intersecting family is a star, provided $n > 2k + 2\sqrt{k \log k}$.

4) I have obtained analogues of the Erdős–Ko–Rado and the Ahlswede–Khachatrian theorems for families of $\{0, \pm 1\}$-vectors.

5) I have resolved Kleitman's problem on the size of the largest family with matching number s in many new cases. For that, we have developed a very powerful averaging method.

6) I have obtained a Hilton–Milner type theorem for the Erdős Matching Conjecture.

7) I studied the chromatic number of random Kneser graphs and showed that it stays almost the same as the chromatic number of Kneser graphs in many scenarios.

8) I have showed that the smallest ϵ-net for range spaces, defined by halfspaces in $\mathbb{R}^d$, has size $\Omega(\frac{d}{\epsilon} \log \frac{1}{\epsilon})$ in the worst case, thus giving a conclusive answer to the question on whether the geometric range spaces are simpler then abstract range spaces.

9) I have obtained sharp lower bounds on the smallest ϵ-nets for range spaces as a function of their shallow cell complexity.

10) I have showed that there are counterexamples to Borsuk's conjecture lying on the spheres of radius arbitrarily close to $1/2$.

11) I have showed that there distance graphs with exponential in the dimension chromatic number and with large girth.

My stability result for the Erdős–Ko–Rado theorem has already found applications in the studies of intersecting families (some of which are presented here), and we believe that it would find more. More generally, the bipartite switching trick that I have introduced seems to be a powerful tool.

The situation is similar with the stability result for the Erdős Matching Conjecture. In a paper with Frankl, we have applied it to advance on an anti-Ramsey type problem. The averaging technique I developed for Kleitman's problem on families with matching number s is very powerful and should find applications elsewhere, in particular, for problems on forbidden subposets.

My results on ϵ-nets have already found an application in statistical learning: in a work with Csikós and Mustafa, we proved that the construction given in the work by muself, Mustafa and Pach allows to resolve a long-standing question on the VC-dimension of a k-fold union $\mathcal{F}^{\cup k}$ of range spaces defined by halfspaces. The notion of shallow cell complexity seems to be very useful in different areas where random sampling is employed, and it is of importance to understand the strength and the limitations of this measure.

Finally the extremal set theory techniques that I introduced in studying distance problems enhance the connections between these two areas and should open the possibilities for some new applications.

Scientific novelty

I have developed the following instruments. I developed the bipartite switching argument for the analysis of intersecting families and especially the phenomenon of stability; moreover, I extended the applications of stability results on intersecting families to different other problems on these families. I developed a set of tools from extremal set theory to deal with families of vectors. I developed an averaging technique for non-uniform families with forbidden configurations and the discharging method for its analysis. I have completed the studies of lower bounds on ϵ-nets in the case of halfspaces and range spaces with bounded shallow cell complexity. I have generalized the Kahn–Kalai transform for Borsuk's problem and introduced new extremal set theory tools in the studies of distance graphs.

The results obtained in the dissertation may be used in the studies of different extremal set theoretic questions, including the questions on families with forbidden configurations and posets with forbidden subposets as well as applied to questions on ϵ-nets in computational geometry and statistical learning and distance problems in discrete geometry.

The propositions for the defence

1) The largest diversity of an intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ is $\binom{n-3}{k-2}$.

2) A typical intersecting family in $\binom{[n]}{k}$ is a star, provided $n > 2k + 2\sqrt{k \log k}$.

3) The largest intersecting family of $\{0, \pm 1\}$-vectors with n coordinates, out of which k are 1's and one is -1, has size $k\binom{n-1}{k}$ for $n \leqslant k^2$ and $k\binom{k^2-1}{k} + \binom{k^2}{k} + \binom{k^2+1}{k} + \ldots + \binom{n-1}{k}$ for $n \geqslant k^2$.

4) For $n = s(m+1) - 2$ the largest family $\mathcal{F} \subset 2^{[n]}$ with no s pairwise disjoint sets has size $\binom{n-1}{m-1} + \sum_{t=m+1}^{n} \binom{n}{t}$.

5) For $n = (u+s-1)(k-1) + s + k$, $u \geqslant s+1$, any family with matching number s and covering number at least $s+1$ has size at most $\binom{n}{k} - \binom{n-s}{k} - \frac{u-s-1}{u}\binom{n-s-k}{k-1}$.

6) The chromatic number of $KG_{n,k}(1/2)$ is at least the chromatic number of $KG_{n,k}$ minus 4, provided $k \geqslant n^{0.51}$.

7) The smallest ϵ-net for range spaces, defined by halfspaces in $\mathbb{R}^d$, has size $\Omega(\frac{d}{\epsilon} \log \frac{1}{\epsilon})$ in the worst case.

8) The upper bounds for the smallest size of an ϵ-net for a range space with fixed shallow cell complexity, obtained by Varadarajan and Chan, Grant, Könemann and Sharpe, are sharp.

9) There are counterexamples to Borsuk's conjecture lying on the spheres of radius arbitrarily close to 1/2.

10) For any t there is $c > 1$, such that for any d there are distance graphs in $\mathbb{R}^d$ with chromatic number at least c^d and with girth at least t.

Methods

In the proofs of the main results we have used the methods of extremal set theory and extremal combinatorics in general, probabilistic method, and methods of discrete geometry. Apart from the standard extremal set theory tools (shifting, the Kruskal–Katona theorem, juntas method, Katona circle, Delta-system method) I have used the methods that I developed and that are described in the section "Scientific novelty".

Theoretical and practical importance

The dissertation is theoretical. The results may be used in other problems coming from extremal set theory, such as questions about intersecting families, families with forbidden subconfigurations and forbidden subposets. More generally the developed tools and results may find further use in extremal combinatorics and coding theory.

The results on ϵ-nets may be used in computational geometry questions, as well as questions in statistical learning and high-dimensional data approximation. Our extremal set theory tools may find future use in the studies of geometric graphs.

Approbation of the work

I have given talks on the topic of the dissertation on the following seminars.

2018: Combinatorics seminar, University of Warwick; Graphes at Lyon, ENS Lyon; Discrete Geometry and Combinatorics Seminar, UCL; Joint DCG–DISOPT seminar, EPFL; Oxford Combinatorics Seminar; Combinatorics seminar, University of Bristol; Seminar on Combinatorics, Games and Optimization, LSE. **2017:** Joint DCG–DISOPT seminar, EPFL; Combinatorics Seminar, University of Birmingham. **2016:** Joint DCG–DISOPT seminar, EPFL; Extremal Combinatorics and Random Structures, MSU; Séminaire de Mathématiques Discrètes, G-SCOP; Structural Learning Seminar, IITP

RAS. **2015:** Joint DCG–DISOPT seminar, EPFL; Seminar on Discrete and Applicable Mathematics LSE; Informal combinatorics seminar, Hebrew University; Research seminar in Combinatorics, Tel Aviv University. **2014 and earlier:** Combinatorics seminar, Free University Berlin; Joint DCG–DISOPT seminar, EPFL; Interdepartmental seminar in discrete mathematics, MIPT.

The results of the dissertation were presented on the following international conferences:

2nd Russan-Hungarian Workshop on Discrete Mathematics, Budapest, June 2018; SoCG, Workshop on Combinatorial Geometry, Budapest, June 2018; Extremal Problems in Combinatorial Geometry, Banff, February 2018; EuroComb'17, Vienna, August 2017; The Second Malta Conference in Graph Theory and Combinatorics, Malta, June 2017; Recent Advances in Extremal Combinatorics Workshop, Sanya, May 2017; 1st Russan-Hungarian Workshop on Discrete Mathematics, Moscow, Apr 2017; MIPT-59, Moscow, November 2016; Journées Graphes et Algorithmes, Paris, November 2016; Oktoberfest in Combinatorial Geometry, Lausanne, October 2016; A New Era of Discrete & Computational Geometry: 30 Years Later, Ascona, June 2016; Moscow Workshop in Discrete Geometry, Moscow, September 2015; Ascension of Combinatorics Conference, Lausanne, May 2015; Sum(m)it 240, Budapest, July 2014; Moscow Workshop on Combinatorics and Number Theory, Moscow, January 2014; EuroComb, Pisa, September 2013; Random structures and algorithms, Poznan', August 2013.

Publications.

The main results of the dissertation were published in 27 papers (21 paper is indexed by Scopus). The list of the publications is given at the end of the synopsis.

The structure of the dissertation

The dissertation consists of the Introduction chapter, 6 chapters in which I present my results, and a bibliography.

The volume of the dissertation is 236 pages, out of which 209 pages of text (excluding the title page, table of contents, list of notations and bibliography). The bibliography consists of 237 items on 19 pages.

SUMMARY OF THE DISSERTATION

In the introduction I present a brief overview of research in extremal combinatorics, as well as a much more detained overview of the results and methods in extremal set theory. I present the main results of the dissertation and its structure.

In Chapter 1, I discuss one of the classical topics of extremal set theory: properties of intersecting families. The following definition is central for the chapter.

Definition 1.1 (The numeration of the definitions, theorems and sections in what follows is the same as in the dissertation). *For integer $t \geqslant 1$, a family $\mathcal{F} \subset 2^{[n]}$ is called* t-intersecting, *if for any $F_1, F_2 \in \mathcal{F}$ we have $|F_1 \cap F_2| \geqslant t$. If $t = 1$, then we call such families* intersecting.

In this chapter, I mostly work with families $\mathcal{F} \subset \binom{[n]}{k}$, that is, *$k$-uniform* families.

I start with giving the statement of the classical theorem of Erdős, Ko and Rado (given in the introduction above) and saying that is tight on a full star. If in a family all sets contain a fixed element, then we call it *trivially intersecting* or a *star*. We call any inclusion-maximal star a *full star*.

The first part of this chapter is devoted to stability for the Erdős–Ko–Rado theorem. Erdős, Ko, and Rado asked, how large can an intersecting family be, provided that it is not trivially (or nontrivially) intersecting? This was answered by Hilton and Milner.

Theorem 1.3 (Hilton, Milner). *Let $n > 2k$ and $\mathcal{F} \subset \binom{[n]}{k}$ be a nontrivially intersecting family. Then $|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.*

The upper bound is attained on the family $\mathcal{H}_k$, where $\mathcal{H}_u$ for integer $u \in [2, k]$ is defined below.

$$\mathcal{H}_u := \left\{A \in \binom{[n]}{k} \ : \ [2, u+1] \subset A\right\} \cup \left\{A \in \binom{[n]}{k} \ : \ 1 \in A, [2, u+1] \cap A \neq \emptyset\right\}.$$

The Hilton–Milner theorem shows that the example for the tightness of the Erdős–Ko–Rado theorem is unique, moreover, that the Erdős–Ko–Rado theorem is stable in the following sense: any intersecting family that is large enough must be a star.

Later, a much stronger stability result was obtained by Frankl in terms of the maximum degree (cf. Theorem 1.4 of the dissertation). For a family $\mathcal{F} \subset 2^{[n]}$, the *degree* $\delta_i(\mathcal{F})$ of an element $i \in [n]$ is the number of sets from $\mathcal{F}$ containing i. We denote by $\Delta(\mathcal{F})$ the largest degree of an element: the maximum of δ_i over $i \in [n]$.

My first contribution that I cover in Chapter 1 starts with the following notion. For a family $\mathcal{F}$, the *diversity* $\gamma(\mathcal{F})$ is the quantity $|\mathcal{F}| - \Delta(\mathcal{F})$. One may think of diversity as of the distance from $\mathcal{F}$ to the closest star. This notion allows us to state a stronger, "dual" version of Theorem 1.4, obtained by myself and Zakharov.

Theorem 1.5. *Let $n > 2k > 0$ and $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family. If $\gamma(\mathcal{F}) \geqslant \binom{n-u-1}{n-k-1}$ for some real $3 \leqslant u \leqslant k$, then*

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} + \binom{n-u-1}{n-k-1} - \binom{n-u-1}{k-1}.$$

The Hilton–Milner theorem, as well as Theorem 1.4, is immediately implied by this theorem. This theorem is sharp for integer values of u, as witnessed by the displayed example above. It is the strongest known stability result for the Erdős–Ko–Rado theorem for large intersecting families, more precisely, for the families of size at least $\binom{n-2}{k-2} + 2\binom{n-3}{k-2}$.

One important insight in Theorem 1.5 is that diversity, rather than the maximum degree, is the correct measure to look at when working with intersecting families. But, more importantly than the result itself, we have developed a simple but very powerful and flexible technique to work with intersecting families, which may be called the *bipartite switching trick* and is based on matchings in regular bipartite graphs. Using it, we managed to obtain a unified proof for several theorems on intersecting and cross-intersecting families.

Definition 1.6. *For positive integer s, two families $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are said to be cross s-intersecting if $|A \cap B| \geqslant s$ holds for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. If $s = 1$ then we call such pairs of families* cross-intersecting*.*

These results are presented in Section 1.3, including the proof of Theorem 1.5.

Next, I discuss the following question: how large could the diversity of an intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ be? This problem was suggested by Katona and addressed by Lemons and Palmer. They found out that for $n > 6k^3$ we have

$\gamma(\mathcal{F}) \leqslant \binom{n-3}{k-2}$, with the equality possible only for $\mathcal{H}_2$ and some of its subfamilies. Recently, Frankl proved that $\gamma(\mathcal{F}) \leqslant \binom{n-3}{k-2}$ for all $n \geqslant 6k^2$, and conjectured that the same holds for $n > 3k$.

I prove the following theorem.

Theorem 1.7. *There exists a constant C, such that for any $n > Ck > 0$ any intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ satisfies $\gamma(\mathcal{F}) \leqslant \binom{n-3}{k-2}$. Moreover, if $\gamma(\mathcal{F}) = \binom{n-3}{k-2}$, then $\mathcal{F}$ is a subfamily of an isomorphic copy of $\mathcal{H}_2$.*

The proof of Theorem 1.7 is presented in Section 1.4. In Section 1.5 I present an interesting counterexample to a very natural conjecture that generalizes Theorem 1.7. The ideas in Section 1.5 rely on the discrete Fourier transform methods in the Analysis of Boolean Functions.

Next, I discuss an application of Theorem 1.5 to the following question: how many different intersecting subfamilies of $\binom{[n]}{k}$ are there?

In an important recent paper, Balogh, Das, Delcourt, Liu, and Sharifzadeh proved that a typical intersecting family is trivial in the following quantitative form. Let $I(n,k)$ denote the *total* number of intersecting families $\mathcal{F} \subset \binom{[n]}{k}$.

Theorem 1.8 (Balogh, Das, Delcourt, Liu, and Sharifzadeh[33]). *If $n \geqslant 3k + 8\log k$ then*

$$I(n,k) = (n + o(1))2^{\binom{n-1}{k-1}},$$

where $o(1) \to 0$ as $k \to \infty$.

For an integer t, denote by $I(n,k,t)$ ($I(n,k,\geqslant t)$) the number of intersecting families with diversity t (at least t). In particular, $I(n,k,\geqslant 1)$ is the number of non-trivial intersecting families. I together with Frankl obtained the following refinement of Theorem 1.8.

Theorem 1.10. *For $n \geqslant 2k + 2 + 2\sqrt{k\log k}$ and $k \to \infty$ we have*

$$\begin{aligned} I(n,k) =&(n + o(1))2^{\binom{n-1}{k-1}},\\ I(n,k,\geqslant 1) =&(1 + o(1))n\binom{n-1}{k}2^{\binom{n-1}{k-1}-\binom{n-k-1}{k-1}}. \end{aligned}$$

This theorem, along with the cross-intersecting version, is proved in Section 1.6.

[33] J. Balogh, S. Das, M. Delcourt, H. Liu, and M. Sharifzadeh, *Intersecting families of dicrete structures are typically trivial*, J. Comb. Theory Ser. A 132 (2015), 224–245.

Next, I discuss product-type inequalities for cross-intersecting families. If $\mathcal{F}$ is intersecting then $\mathcal{A} := \mathcal{F}$, $\mathcal{B} := \mathcal{F}$ are cross-intersecting. Therefore the following result implies the Erdős–Ko–Rado theorem.

Theorem 1.11 (Pyber[34]). *Suppose that $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ are cross-intersecting and $n \geqslant 2k$. Then*

$$|\mathcal{A}||\mathcal{B}| \leqslant \binom{n-1}{k-1}^2 \quad \textit{holds.}$$

My contribution (joint with Frankl) in this direction is two-fold. First we provide a very short proof of the theorem above. Then we use the ideas of this proof and some counting based on the Kruskal–Katona Theorem to obtain the following sharper, best possible bounds.

Theorem 1.12. *Let $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ be cross-intersecting, $n > 2k > 0$ and suppose $|\mathcal{A}| \leqslant \binom{n-1}{k-1} \leqslant |\mathcal{B}|$ and $\cap_{B\in\mathcal{B}} B = \emptyset$. Then*

$$|\mathcal{A}||\mathcal{B}| \leqslant \left(\binom{n-1}{k-1} + 1\right)\left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1}\right) \quad \textit{holds.}$$

Theorem 1.13. *Let $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$ be cross-intersecting, $n \geqslant 2k > 0$ and suppose that $|\mathcal{B}| \geqslant \binom{n-1}{k-1} + \binom{n-i}{k-i+1}$ holds for some $3 \leqslant i \leqslant k+1$. Then*

$$|\mathcal{A}||\mathcal{B}| \leqslant \left(\binom{n-1}{k-1} + \binom{n-i}{k-i+1}\right)\left(\binom{n-1}{k-1} - \binom{n-i}{k-1}\right).$$

These results are proved in Section 1.7.

Next, I discuss cross s-intersecting families and cross s-intersecting families that are additionally t-intersecting. The question I address is what is the maximum of $|\mathcal{A}| + |\mathcal{B}|$, where $\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}$, possess the properties mentioned above and, moreover, are non-empty. The cross s-intersecting problem is resolved completely, and if the families are additionally t-intersecting, then we have an exact solution for large n. The statements of the results could be found in Sections 1.1.5 and 1.1.6, and their proofs in Sections 1.8 and 1.9, respectively.

In Chapter 2 I study the collections, or families, of vectors in $\{-1, 0, 1\}^n$, which are for shorthand called $\{0, \pm 1\}$-*vectors*. I start with explaining the connection between the intersection-type results on families and the problems in discrete geometry. (The easy first step is to think of sets as $\{0, 1\}$-vectors in $\mathbb{R}^n$.) This connection is especially fruitful in the case of the Frankl–Wilson theorem.

[34] L. Pyber, *A new generalization of the Erdős–Ko–Rado theorem*, J. Comb. Theory Ser. A 43 (1986), 85–90.

Theorem 2.1. *If n is a prime power and $\mathcal{F} \subset \binom{[4n-1]}{2n-1}$ satisfies $|F_1 \cap F_2| \neq n-1$ for any $F_1, F_2 \in \mathcal{F}$, then the size of $\mathcal{F}$ is at most $\binom{4n-1}{n-1}$.*

Note that $\binom{4n-1}{n-1}$ is exponentially smaller than $\binom{4n-1}{2n-1}$. Due to that, this theorem has several important implications for the chromatic number of the space, as well as some results in Euclidean Ramsey theory. Kahn and Kalai gave it a twist to deduce counterexamples to the famous Borsuk conjecture.

Raigorodskii[35] succeeded in improving the bounds in the aforementioned geometric problems by enlarging the scope of vectors from $\{0,1\}$-vectors to $\{0,\pm1\}$-vectors. This motivated a more systematic investigation of extremal questions for families of $\{0,\pm1\}$-vectors.

More precisely, I cover two types of results. The results of the first type generalize the Erdős–Ko–Rado theorem to the families of $\{0,\pm1\}$-vectors with a fixed number of $+1$-coordinates and -1-coordinates. The results of the second type deal with the families of $\{0,\pm1\}$-vectors with a fixed number of non-zero coordinates, that is, of fixed length.

I start with vectors with a fixed number of $+1$'s and -1's. The next definition gives the class of vectors I deal with.

Definition 2.2. *For $0 \leqslant l, k < n$ define $\mathcal{V}(n,k,l) \subset \mathbb{R}^n$ to be the set of all $\{0,\pm1\}$-vectors having exactly k $+1$-coordinates and l -1-coordinates.*

Note that
$$|\mathcal{V}(n,k,l)| = \binom{n}{k}\binom{n-k}{l}.$$
With this notation, families of k-sets are just subsets of $\mathcal{V}(n,k,0)$.

For vectors $\mathbf{v}, \mathbf{w}$ their *scalar product*, is denoted by $\langle \mathbf{v}, \mathbf{w} \rangle$. If both $\mathbf{v}$ and $\mathbf{w}$ are $\{0,1\}$-vectors, then their scalar product is non-negative with $\langle \mathbf{v}(F), \mathbf{v}(G) \rangle = 0$ iff $F \cap G = \emptyset$.

For $n \geqslant 2k$ the minimum possible scalar product in $\mathcal{V}(n,k,l)$ is $-2l$. Such a scalar product is achieved on a pair of vectors iff the -1's in each of them stand on the positions of $+1$'s in the other, and no two $+1$'s stand on the same coordinate position.

Definition 2.3. *A family $\mathcal{V} \subset \mathcal{V}(n,k,l)$ of vectors is called* intersecting *if the scalar product of any two vectors in $\mathcal{V}$ is strictly larger than the minimum scalar product in $\mathcal{V}(n,k,l)$.*

[35] A.M. Raigorodskii, *The Borsuk problem and the chromatic numbers of some metric spaces*, Russian Math. Surveys 56 (2001), N1, 103–139.

By analogy with the Erdős–Ko–Rado Theorem, define

$$m(n,k,l) := \{\max|\mathcal{V}| : \mathcal{V} \subset \mathcal{V}(n,k,l), \mathcal{V} \text{ is intersecting}\}.$$

My first contribution in this chapter is described in two theorems, which together give the exact value of $m(n,k,1)$ for all n,k. One surprising fact is that the situation is very different from the case $l=0$, that is, the Erdős–Ko–Rado Theorem. Namely, while for $n \leqslant k^2$ the Erdős–Ko–Rado-type construction is optimal, it is no longer optimal for $n > k^2$.

Theorem 2.5. *For $2k \leqslant n \leqslant k^2$*

$$m(n,k,1) = k\binom{n-1}{k} \qquad \textit{holds.}$$

The proof of this theorem for $n \geqslant 3k$ is given in Section 2.2.5, and for $2k \leqslant n \leqslant 3k-1$ in Section 2.4 as a consequence of a more general result (Theorem 2.8).

Theorem 2.6. *For $n \geqslant k^2$ one has*

$$m(n+1,k,1) = m(n,k,1) + \binom{n}{k}.$$

The proof of Theorem 2.6 is presented in Section 2.2, and Section 2.2 contains the necessary preparations.

Then we discuss some partial results for $l > 1$ and explain how to deal with the easy cases $k = l$ or $n \leqslant 2k$.

Next I give results on $\{0,\pm1\}$-vectors of fixed length. Denote by $\mathcal{L}_k$ the family of all vectors $\mathbf{v}$ from $\{0,\pm1\}^n$ such that $\langle \mathbf{v},\mathbf{v}\rangle = k$. Note that $|\mathcal{L}_k| = 2^k\binom{n}{k}$. We are interested in the quantity below.

$$F(n,k,l) := \max\{|\mathcal{V}| : \mathcal{V} \subset \mathcal{L}_k, \forall\, \mathbf{v},\mathbf{w} \in \mathcal{V} \;\; \langle \mathbf{v},\mathbf{w}\rangle \geqslant l\}.$$

Recall the following theorem of Katona:

Theorem (Katona[36]). *Let $n > s > 0$ be fixed integers. If $\mathcal{U} \subset 2^{[n]}$ is a family of sets such that for any $U, V \in \mathcal{U}$ we have $|U \cup V| \leqslant s$ then*

$$|\mathcal{U}| \leqslant f(n,s) := \begin{cases} \displaystyle\sum_{i=0}^{s/2}\binom{n}{i} & \textit{if } s \textit{ is even,} \\ \displaystyle 2\sum_{i=0}^{(s-1)/2}\binom{n-1}{i} & \textit{if } s \textit{ is odd.} \end{cases}$$

[36] G.O.H. Katona, *Intersection theorems for systems of finite sets*, Acta Math. Acad. Sci. Hung. 15 (1964), 329–337.

Moreover, for $n \geqslant s+2$ the equality is attained only for the following families. If s is even, than it is the family $\mathcal{U}^s$ of all sets of size at most $s/2$. If s is odd, then it is one of the families U^s_j of all sets that intersect $[n]-\{j\}$ in at most $(s-1)/2$ elements, where $1 \leqslant j \leqslant n$.

The main result in this direction is the following theorem, which determines $F(n,k,l)$ for all k,l and sufficiently large n and which shows the connection of $F(n,k,l)$ with the above stated theorem of Katona.

Theorem 2.10. *For any k and $n \geqslant n_0(k)$ we have*

$$1. \qquad F(n,k,l) = \binom{n-l}{k-l} \qquad \text{for } \ 0 \leqslant l \leqslant k.$$

$$2. \qquad F(n,k,-l) = f(k,l)\binom{n}{k} \quad \text{for } \ 0 \leqslant l \leqslant k.$$

Using the same technique, we may extend the result of part 2 of Theorem 2.10 in the following way:

Theorem 2.11. *Let $\mathcal{V} \subset \mathcal{L}_k$ be the set of vectors such that for any $\mathbf{v}, \mathbf{w} \in \mathcal{V}$ we have $\langle \mathbf{v}, \mathbf{w}\rangle \neq -l-1$ for some $0 \leqslant l < k$. Then we have*

$$\max_{\mathcal{V}} |\mathcal{V}| = f(k,l)\binom{n}{k} + O(n^{k-1}).$$

These results, along with some simple observations concerning $F(n,k,l)$ and the complete solution for the case $k=3$, are presented in Section 2.5.

In Chapter 3, I discuss families with no s pairwise disjoint sets. The maximum number of pairwise disjoint members of a family $\mathcal{F}$ is denoted by $\nu(\mathcal{F})$ and called the *matching number* of $\mathcal{F}$. Here are the quantities that I study in this chapter.

Definition 3.2. *For $n \geqslant s \geqslant 2$ define*

$$e(n,s) := \max\big\{|\mathcal{F}| : \mathcal{F} \subset 2^{[n]}, \nu(\mathcal{F}) < s\big\}.$$

Similarly, for positive integers $n,k,s \geqslant 2$, $n \geqslant ks$ define

$$e_k(n,s) := \max\Big\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k}, \nu(\mathcal{F}) < s\Big\}.$$

Families with matching number 1 are intersecting and vice versa. Thus, $e_k(n,2)$ is determined in the Erdős–Ko–Rado theorem.

For $m = \lceil \frac{n+1}{s} \rceil$ the family

$$\binom{[n]}{\geqslant m} := \{H \subset [n] : |H| \geqslant m\}$$

does not contain s pairwise disjoint sets. Erdős conjectured that for $n = sm - 1$ one cannot do any better. Half a century ago Kleitman proved this conjecture and determined $e(sm, s)$ as well.

Theorem (Kleitman)**.** *Let $s \geqslant 2, m \geqslant 1$ be integers. Then the following holds.*

$$\text{For } n = sm - 1, \quad \text{we have} \quad e(n,s) = \sum_{m \leqslant t \leqslant n} \binom{n}{t},$$
$$\text{for } n = sm, \quad \text{we have} \quad e(n,s) = \frac{s-1}{s}\binom{n}{m} + \sum_{m+1 \leqslant t \leqslant n} \binom{n}{t}.$$

I first make a general conjecture.

Definition 3.3. *Let $n = sm + s - l$, $0 < l \leqslant s$. Set*

$$\mathcal{P}(s,m,l) := \left\{P \subset 2^{[n]} : |P| + |P \cap [l-1]| \geqslant m+1\right\},$$

In the dissertation, it is shown that $\nu(\mathcal{P}(s,m,l)) < s$.

Conjecture. *Suppose that $s \geqslant 2, m \geqslant 1$, and $n = sm + s - l$ for some $0 < l \leqslant \lceil \frac{s}{2} \rceil$. Then*

$$e(sm + s - l, s) = |\mathcal{P}(s,m,l)| \qquad \textit{holds.}$$

The main result in this chapter (joint with Frankl) is the proof of Conjecture above in a relatively wide range.

Theorem 3.4. $e(sm + s - l, s) = |\mathcal{P}(s,m,l)|$ *holds for*

(i) $l = 2$ *and* $s \geqslant 3$, (ii) $m = 1$,
(iii) $s \geqslant lm + 3l + 3$.

The proof of (i) for $s \geqslant 5$ is given in Section 3.2.4. The proof of (ii) and (iii) is given in Section 3.2.5. Contrary to the intuition, the problem gets easier as s becomes larger, and thus the proof for $s = 3, 4$ is more intricate. It is given in Sections 3.3.2 and 3.3.3, respectively.

The proof for $s = 3, 4$ is based on a non-trivial averaging technique somewhat in the spirit of Katona's circle method: we choose a certain configuration of sets, show that the intersection of a family satisfying the conditions of Theorem 3.4 with *each* such configuration cannot be too large and then average over all such configurations. However, the configuration is quite complicated, the sets in the configuration actually have weights, and, in order to bound the weighted intersection of the family with each configuration, we use some kind of discharging method.

In Section 3.4, we analyze this method and obtain interesting new inequalities concerning the structure of the families with small matching number.

The problems of determining $e(n,s)$ and $e_k(n,s)$ are, in fact, closely interconnected. In the proof of Theorem 3.4 we use some a stability result for the so-called Erdős Matching Conjecture, which is interesting in its own right.

There are several natural examples of a family $\mathcal{A} \subset \binom{[n]}{k}$ satisfying $\nu(\mathcal{A}) = s$ for $n \geqslant (s+1)k$. Let us define the families $\mathcal{A}_i^{(k)}(n,s)$:

$$\mathcal{A}_i^{(k)}(n,s) := \left\{A \in \binom{[n]}{k} : |A \cap [(s+1)i-1]| \geqslant i\right\}, \quad 1 \leqslant i \leqslant k.$$

Conjecture (Erdős Matching Conjecture). *For $n \geqslant (s+1)k$*

$$e_k(n,s+1) = \max\left\{|\mathcal{A}_1^{(k)}(n,s)|, |\mathcal{A}_k^{(k)}(n,s)|\right\}.$$

The Erdős Mathichng Conjecture is known to be true for $k \leqslant 3$, and Frankl showed that

$$e_k(n,s+1) = \binom{n}{k} - \binom{n-s}{k} \qquad \text{is proven for} \quad n \geqslant (2s+1)k - s.$$

Recently, in a joint work with P. Frankl, we have proved that

The equality above is true for any $s \geqslant s_0$ and $n \geqslant \frac{5}{3}sk - \frac{2}{3}s$.

In the case $s = 1$ (that is, the case of the Erdős–Ko–Rado Theorem) one has a very useful stability theorem due to Hilton and Milner. In a work with Frankl, I proved its (weak) analogue for the Erdős Matching Conjecture.

Theorem 3.5. *Let $n = (u+s-1)(k-1)+s+k$, $u \geqslant s+1$. Then for any family $\mathcal{G} \subset \binom{[n]}{k}$ with $\nu(\mathcal{G}) = s$ and $\tau(\mathcal{G}) \geqslant s+1$ we have*

$$|\mathcal{G}| \leqslant \binom{n}{k} - \binom{n-s}{k} - \frac{u-s-1}{u}\binom{n-s-k}{k-1}.$$

It is proved in Section 3.2.3. I note that a stronger statement was proven in another paper due to myself and Frankl.

In Chapter 4, I discuss the results on random Kneser graphs and hypergraphs. Kneser graphs and hypergraphs are very popular and well-studied objects in combinatorics. Fix some positive integers n, k, r, where $r \geqslant 2$. The set of vertices of the Kneser r-graph $KG^r_{n,k}$ is the set of all k-element subsets of $[n]$, denoted by $\binom{[n]}{k}$. The set of edges of $KG^r_{n,k}$ consists of all r-tuples of pairwise disjoint subsets. (When dealing with the graph case $r = 2$, we omit the superscript in the notation of Kneser graphs.) For a hypergraph $\mathcal{H}$ we denote by $\chi(\mathcal{H})$ its *chromatic number*, that is, the minimum number χ such that there exists a coloring of vertices $\mathcal{H}$ into χ colors that leaves no edge of $\mathcal{H}$ monochromatic.

Kneser graphs earned their name from M. Kneser, who investigated them in the 50s. He showed that $\chi(KG_{n,k}) \leqslant n-2k+2$ and conjectured that this bound is tight. This conjecture (or rather its resolution) played a very important role in combinatorics. It was confirmed by L. Lovász[37], who, in order to resolve it, introduced tools from algebraic topology to combinatorics. After Lovász' paper, there was a burst of activity around Kneser graphs, and I give a detailed account of it in the introduction to Chapter 4.

Schrijver studied induced subgraphs $SG_{n,k}$ of $KG_{n,k}$ constructed on the family of all k-element *stable sets* of the cycle C_n. In other words, the underlying family contains all k-element sets that do not have two cyclically consecutive elements of $[n]$. He showed that they have the same chromatic number as $KG_{n,k}$: $\chi(SG_{n,k}) = n - 2k + 2$.

To state results on random Kneser (hyper)graphs, we need the following definition. For a hypergraph $\mathcal{H}$ and a real number p, $0 < p < 1$, define the *random hypergraph* $\mathcal{H}(p)$ as follows: the set of vertices of $\mathcal{H}(p)$ coincides with that of $\mathcal{H}$, and the set of edges of $\mathcal{H}(p)$ is a subset of the set of edges of $\mathcal{H}$, with each edge from $\mathcal{H}$ taken independently and with probability p. The results on random graphs and hypergraphs, roughly speaking, tell us how does a *typical* subgraph of a given (hyper)graph that contains a p-fraction of edges behave with respect to a given property.

One class of questions that is particularly relevant for this paper deals with transference results. In general, we speak of transference if a certain combinatorial result holds with no changes in the random setting. Studying this phenomenon in the context of the independence number and the chromatic number of generalized Kneser graphs was suggested by Bogolyubskiy, Gusev, Pyaderkin and Raigorodskii.

In this chapter, I present two approaches to provide lower bounds on the chromatic number of random Kneser graphs. I found the first one earlier, and it is of topological flavour. The second approach is purely combinatorial and allows us to significantly improve all previously known bounds on the chromatic numbers

[37] L. Lovász, *Kneser's conjecture, chromatic number, and homotopy*, J. Comb. Theory Ser. A 25 (1978), N3, 319–324.

in the most interesting cases: for random subgraphs of (complete) Kneser and Schrijver graphs and Kneser hypergraphs. Here is the main theorem of the chapter, which is proved in Section 4.3.

Theorem 4.2. *Let $p \in (0,1)$ and $r \in \mathbb{N}, r \geqslant 2$, be fixed.*
If $r = 2^q$ for some $q \in \mathbb{N}$ then a.a.s.

$$\textbf{(}\boldsymbol{l=1,\ r=2^q}\textbf{)}\quad \chi(KG^r_{n,k})(p) \geqslant \chi(KG^r_{n,k+1}) \text{ if } n - rk \ll n^{r/(r+1)} \log^{-1/(r+1)} n.$$

If $r = 2$ then a.a.s.

$$\textbf{(fixed } \boldsymbol{l,\ r=2}\textbf{)}\quad \chi(KG_{n,k})(p) \geqslant \chi(KG_{n,k+l}) \text{ if } l \text{ is fixed and } k \gg (n \log n)^{1/l};$$
$$\textbf{(fixed } \boldsymbol{k,\ r=2}\textbf{)}\quad \chi(KG_{n,k})(p) \geqslant \chi(KG_{n,k+l}) \text{ if } k \text{ is fixed and } l \gg (n \log n)^{1/k}.$$

If $r = 3$ then a.a.s.

$$\textbf{(fixed } \boldsymbol{l,\ r=3}\textbf{)}\ \chi(KG^3_{n,k})(p) \geqslant \chi(KG^3_{n,k+l}) \text{ if } l \text{ is fixed and } k \gg \log^{1/(3l-4)} n;$$
$$\textbf{(fixed } \boldsymbol{k,\ r=3}\textbf{)}\ \chi(KG^3_{n,k})(p) \geqslant \chi(KG^3_{n,k+l}) \text{ if } k \text{ is fixed and } l \gg \log^{2/(6k-11)} n.$$

If $r > 3$ then a.a.s.

$$\textbf{(fixed } \boldsymbol{l,\ r>3}\textbf{)}\ \chi(KG^r_{n,k})(p) \geqslant \chi(KG^r_{n,k+l}) \text{ if } l \text{ is fixed and } k \gg \log^{\frac{1}{r(l-2)-1}} n;$$
$$\textbf{(fixed } \boldsymbol{k,\ r>3}\textbf{)}\ \chi(KG^r_{n,k})(p) \geqslant \chi(KG^r_{n,k+l}) \text{ if } k \text{ is fixed and } l \gg \log^{\frac{1}{r(k-1)-\frac{2r-1}{r-1}}} n.$$

Probably, the most interesting results in Theorem 4.2 are for $r \geqslant 3$. They are much stronger than the previous results of Alishahi and Hajabolhassan[38] and guarantee that the chromatic number of $KG^r_{n,k}(p)$ drops by no more than a small additive term for already for polylogarithmic k (this was known before for polynomial k).

In Chapter 5 I discuss my results on ϵ-nets. Let X be a finite set and let $\mathcal{R}$ be a system of subsets of an underlying set containing X. In computational geometry, the pair $(X, \mathcal{R})$ is usually called a *range space*. A subset $X' \subseteq X$ is called an *ϵ-net* for $(X, \mathcal{R})$ if $X' \cap R \neq \emptyset$ for every $R \in \mathcal{R}$ with at least $\epsilon|X|$ elements. The use of small-sized ϵ-nets in geometrically defined range spaces has become a standard technique in discrete and computational geometry, with many combinatorial and algorithmic consequences.

For any subset $Y \subseteq X$, define the *projection* of $\mathcal{R}$ on Y to be the set system

$$\mathcal{R}|_Y := \big\{Y \cap R : R \in \mathcal{R}\big\}.$$

[38] M. Alishahi and H. Hajiabolhassan, *Chromatic Number of Random Kneser Hypergraphs*, J. Comb. Theory Ser. A 154 (2018), 1–20.

The *Vapnik-Chervonenkis dimension* or, in short, the *VC-dimension* of the range space $(X, \mathcal{R})$ is the minimum integer d such that $|\mathcal{R}|_Y| < 2^{|R|}$ for any subset $Y \subseteq X$ with $|Y| > d$. According to the Vapnik–Chervonenkis–Sauer–Shelah lemma, for any range space $(X, \mathcal{R})$ whose VC-dimension is at most d and for any subset $Y \subseteq X$, we have $|\mathcal{R}|_Y| = O(|Y|^d)$.

Haussler and Welzl showed that if the VC-dimension of a range space $(X, \mathcal{R})$ is at most d, then by picking a random sample of size $\Theta(\frac{d}{\epsilon} \log \frac{1}{\epsilon})$, we obtain an ϵ-net with positive probability. It was by Komlos, Woeginger an Pach that this bound is essentially tight for random range spaces.

The effectiveness of ϵ-net theory in geometry derives from the fact that most "geometrically defined" range spaces $(X, \mathcal{R})$ arising in applications have bounded VC-dimension and, hence, satisfy the preconditions of the ϵ-net theorem.

One of the range spaces I discuss is as follows: the vertices are points in $\mathbb{R}^d$, and the ranges are formed by all possible intersections of the point set with halfspaces. My first contribution is that, generalizing the result of Pach and Tardos in $\mathbb{R}^4$, in a work with Mustafa and Pach I showed that the Haussler–Welzl upper bound for ϵ-nets is tight for the range space described above.

Theorem 5.3. *For any integer $d \geqslant 4$, real $\epsilon > 0$ and any sufficiently large integer $n \geq n_0(\epsilon)$, there exist primal range spaces $(X, \mathcal{F})$ induced by n-element point sets X and collections of half-spaces $\mathcal{F}$ in $\mathbb{R}^d$ such that the size of every ϵ-net for $(X, \mathcal{F})$ is at least $\frac{\lfloor d/4 \rfloor}{9\epsilon} \log \frac{1}{\epsilon}$.*

This theorem is proved in Sections 5.2 and 5.3.

Following the work of Clarkson and Varadarajan, it has been gradually realized that if one replaces the condition that the range space $(X, \mathcal{R})$ has bounded VC-dimension by a more refined combinatorial property, one can prove the existence of ϵ-nets of size $o(\frac{1}{\epsilon} \log \frac{1}{\epsilon})$. To formulate this property, we need to introduce some terminology.

Given a function $\varphi : \mathbb{N} \to \mathbb{R}^+$, we say that the primal range space $(X, \mathcal{R})$ has *shallow-cell complexity* φ if there exists a constant $c = c(\mathcal{R}) > 0$ such that, for every $Y \subseteq X$ and for every positive integer l, the number of at most l-element sets in $\mathcal{R}|_Y$ is $O\big(|Y| \cdot \varphi(|Y|) \cdot l^c\big)$. Note that if the VC-dimension of $(X, \mathcal{R})$ is d, then for every $Y \subseteq X$, the number of elements of the projection of the set system $\mathcal{R}$ to Y satisfies $|\mathcal{R}|_Y| = O(|Y|^d)$. However, the condition that $(X, \mathcal{R})$ has *shallow-cell complexity* φ for some function $\varphi(n) = O(n^{d'}), 0 < d' < d - 1$ and some constant $c = c(\mathcal{R})$, implies not only that $|\mathcal{R}|_Y| = O(|Y|^{1+d'+c})$, but it reveals some nontrivial finer details about the distribution of the sizes of the smaller members of $\mathcal{R}|_Y$.

Many of the geometrically defined range spaces turn out to have low shallow-cell complexity. For instance, the primal range spaces induced by containment of points in disks in $\mathbb{R}^2$ or half-spaces in $\mathbb{R}^3$ have shallow-cell complexity $\varphi(n) = O(1)$.

A series of elegant results of Aronov, Ezra and Sharir, Varadarajan, and Chan, Grant, Könemann and Sharpe illustrate that if the shallow-cell complexity of a set system is $\varphi(n) = o(n)$, then it permits smaller ϵ-nets than what is guaranteed by the ϵ-net theorem of Haussler and Welzl.

Theorem 5.1. *Let $(X, \mathcal{R})$ be a range space with shallow-cell complexity φ, where $\varphi(n) = O(n^d)$ for some constant d. Then, for every $\epsilon > 0$, it has an ϵ-net of size $O\big(\frac{1}{\epsilon}\log\varphi(\frac{1}{\epsilon})\big)$, where the constant hidden in the O-notation depends on d.*

In a work with Mustafa and Pach, I showed that the bound in Theorem 5.1 cannot be improved.

Definition 5.5. *A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called* submultiplicative *if there exists an $\alpha, 0 < \alpha < 1$ such that*

1) $\varphi^\alpha(x) \leqslant \varphi(x^\alpha)$ for all sufficiently large $x \in \mathbb{R}^+$, and

2) $\varphi(xy) \leqslant \varphi(x)\varphi(y)$ for all sufficiently large $x, y \in \mathbb{R}^+$.

Theorem 5.6. Let d be a fixed positive integer and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be any submultiplicative function with $\varphi(n) = O(n^d)$. Then, for any $\epsilon > 0$ there exist range spaces $(X, \mathcal{F})$ that have
(i) shallow-cell complexity φ, and for which
(ii) the size of any ϵ-net is at least $\Omega(\frac{1}{\epsilon}\log\varphi(\frac{1}{\epsilon}))$.

This theorem is proved in Section 5.4.

In Chapter 6 I discuss applications of extremal set theory results to problems on geometric graphs. First I discuss the classical Borsuk partition problem. In 1933 K. Borsuk[39] posed the following question: *is it true that any set $\Omega \subset \mathbb{R}^d$ having diameter 1 can be divided into some parts $\Omega_1, \ldots, \Omega_{d+1}$ whose diameters are strictly smaller than 1?* Here by the *diameter* of a set Ω we mean the quantity

$$\operatorname{diam}\Omega := \sup_{\mathbf{x},\mathbf{y}\in\Omega} |\mathbf{x} - \mathbf{y}|,$$

where, in turn, $|\mathbf{x} - \mathbf{y}|$ denotes the standard Euclidean distance between vectors.

[39] K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. 20 (1933), 177–190.

Let us introduce the following notation. By $f(\Omega)$ we denote the value

$$f(\Omega) := \min\{f: \ \Omega = \Omega_1 \cup \ldots \cup \Omega_f, \ \ \forall\, i \ \ \operatorname{diam} \Omega_i < \operatorname{diam} \Omega\}.$$

Furthermore,

$$f(d) := \max_{\Omega \subset \mathbb{R}^d, \ \operatorname{diam} \Omega = 1} f(\Omega),$$

i.e., $f(d)$ is the minimum number of parts of smaller diameter, into which an arbitrary set of diameter 1 in $\mathbb{R}^d$ can be divided. In these terms, Borsuk's question is as follows: *is it true that always* $f(d) = d+1$? The positive answer on this question is usually called "Borsuk's conjecture".

This conjecture was shown to be false by Kahn and Kalai[40], and the bounds on $f(d)$ are exponential in d have the form $c^{\sqrt{d}} \leqslant f(d) \leqslant C^d$, where $c, C > 1$ are some constants. The colorful history of Borsuk's conjecture is exhibited in numerous books and survey papers. We refer the reader to the books of Raigorodskii and Brass, Moser and Pach.

A careful analysis of all the known counterexamples to Borsuk's conjecture shows that they are always finite sets of points in $\mathbb{R}^d$ lying on spheres whose radii are close to $\frac{1}{\sqrt{2}}$. This is quite natural, since, by Jung's theorem, any set in $\mathbb{R}^d$ having diameter 1 can be covered by a ball of radius $\sqrt{\frac{d}{2d+2}} \sim \frac{1}{\sqrt{2}}$, and the intuition is that in order to get a counterexample, we have to take a set with as big covering ball as possible. The main result in this section is the following theorem which completely breaks such intuition.

Theorem 6.1. *Let $S_r^{d-1} \subset \mathbb{R}^d$ be the sphere of radius r with center at the origin. For any $r > \frac{1}{2}$, there exists a $d_0 = d_0(r)$ such that for every $d \geqslant d_0$, one can find a set $\Omega \subset S_r^{d-1}$ which has diameter 1 and does not admit a partition into $d+1$ parts of smaller diameter.*

The proof and some stronger results are given in Section 6.2.

Next, I discuss a question related to the chromatic number of the space: distance graphs with high girth. We say that $G = (V, E)$ is an *distance graph* in $\mathbb{R}^n$, if V is a subset of $\mathbb{R}^n$ and

$$E \subseteq \{\{\mathbf{x}, \mathbf{y}\} : \ \mathbf{x}, \mathbf{y} \in V, \ \ |\mathbf{x} - \mathbf{y}| = 1\}.$$

Such graphs arise naturally in the context of the problem of finding the chromatic number of the space. This famous question was posed by Nelson in 1950:

[40] J. Kahn and G. Kalai, *A counterexample to Borsuk's conjecture*, Bull. Amer. Math. Soc. 29 (1993), 60–62.

what is the minimum number $\chi(\mathbb{R}^2)$ of colors needed to color all points of the plane so that no two points at distance one receive the same color? One may ask the same question for higher-dimensional spaces. It is known that $\chi(\mathbb{R}^d)$ grows exponentially with d, and the best bounds are due to Raigorodskii from below, and Larman and Rogers from above.

Due to result of de Bruijn and P. Erdős, the chromatic number of the space $\mathbb{R}^n$ is equal to the maximum of the chromatic numbers of (finite) distance graphs in $\mathbb{R}^n$.

The motivation for the studies presented in this part of Chapter 6 comes from the following classical question. Can we construct graphs with arbitrarily large chromatic number and arbitrary girth (the length of the shortest cycle)? The positive answer to this question for general graphs was given by P. Erdős, and later Lovász came up with an explicit construction.

Similar questions for distance graphs were studied before. O'Donnell gave examples of graphs with high girth and chromatic number 4 on the plane, and Demekhin, Raigorodskii and Rubanov[41] for any k managed to construct distance graphs in $\mathbb{R}^d$ with no odd cycles of length shorter than k, such that their chromatic number is at least c^d for some $c > 1$. They asked, whether it is possible to do construct an analogous family of graphs without *any* cycles of length k. I answered this question.

Theorem 6.2. *For any fixed $k \geqslant 3$ there exists $c > 1$, such that for any $d \geqslant 1$ in $\mathbb{R}^d$ there exists a distance graph with girth at least k and with chromatic number at least c^d.*

This theorem is proved in Section 6.3.

[41] E.E. Demechin, A.M. Raigorodskii, and O.I. Rubanov, *Distance graphs having large chromatic numbers and containing no cliques or cycles of a given size*, Sbornik: Mathematics 204 (2013), N4, 508–538.

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Publications of the author on the subject of the dissertation

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