Federal State Autonomous Educational Institution of Higher Education "Moscow Institute of Physics and Technology (National Research University)"

APPROVED
Head of the Phystech School of Applied Mathematics and Informatics
A.M. Raygorodskiy

Work program of the course (training module)

course:	Introduction to Discrete Geometry/Введение в дискретную геометрию
major:	Applied Mathematics and Informatics
specialization:	Advanced Methods of Modern Combinatorics/Продвинутые методы современной комбинаторики
	Phystech School of Applied Mathematics and Informatics
	Chair of Discrete Mathematics
term:	1
qualification:	Master

Semesters, forms of interim assessment:

1 (fall) - Grading test

2 (spring) - Exam

Academic hours: 120 AH in total, including:
lectures: 60 AH.
seminars: 60 AH.
laboratory practical: 0 AH.

Independent work: 75 AH.

Exam preparation: 30 AH.

In total: 225 AH, credits in total: 5

Author of the program: A.A. Polyanskiy, candidate of physics and mathematical sciences, associate professor

The program was discussed at the Chair of Discrete Mathematics 05.03.2020

Annotation

The course will focus on classical and modern methods and results in discrete geometry. We start with classical statements from the field of convex and discrete geometry (theorems of Radon, Carathéodory, Helly, Tverberg), then we touch on their colored, fractional and combinatorial versions (epsilon networks). Then we move on to questions at the junction with the geometry of high dimensions (John's theorem, Dvoretsky's theorem, and others)

1. Study objective

Purpose of the course

mastering an advanced discrete geometry course.

Tasks of the course

- students mastering basic knowledge (concepts, concepts, methods and models) in the field of discrete geometry;
- acquisition of theoretical knowledge and practical skills in the field of discrete geometry;
- providing advice and assistance to students in conducting their own theoretical research in the field of discrete geometry.

2. List of the planned results of the course (training module), correlated with the planned results of the mastering the educational program

Mastering the discipline is aimed at the formation of the following competencies:

Code and the name of the competence	Competency indicators
UC-2 Able to manage a project through all stages of its life cycle	UC-2.4 Publicly present the project results (or results of its stages) via reports, articles, presentations at scientific conferences, seminars, and similar events
Gen.Pro.C-1 Address current challenges in fundamental and applied mathematics	Gen.Pro.C-1.1 Apply fundamental scientific knowledge, new scientific principles, and research methods in applied mathematics and computer science
Gen.Pro.C-3 Develop mathematical models and conduct their analysis in the processes of professional problem-solving	Gen.Pro.C-3.2 Employ research methods to solve new problems, and apply knowledge from various science and technology fields
Pro.C-1 Become part of a professional community and conduct local research under scientific guidance using methods specific to a particular professional setting	Pro.C-1.1 Apply principles of scientific work, methods of data collection and analysis, ways of argumentation; prepare scientific reviews, publications, abstracts, and bibliographies on research topics in Russian and English
	Pro.C-1.3 Use practical knowledge of scientific argumentation when analyzing a research subject area
Pro.C-3 Participate in scholarly discussions, make speeches and presentations (oral, written, and online) on scientific topics, present research materials, proofread, edit, reference scientific works	Pro.C-3.1 Learn the basics of scholarly discussion and the forms of verbal scientific communication

3. List of the planned results of the course (training module)

As a result of studying the course the student should:

know:

fundamental concepts, laws, theories of discrete geometry;

current problems of the corresponding sections of discrete geometry;

concepts, axioms, methods of proof and proof of the main theorems in the sections included in the basic part of the cycle;

basic properties of the corresponding mathematical objects;

analytical and numerical approaches and methods for solving typical applied problems of discrete geometry.

be able to:

understand the task;
use your knowledge to solve fundamental and applied problems of discrete geometry;
evaluate the correctness of the problem statements;
strictly prove or disprove the statement;
independently find algorithms for solving problems, including non-standard ones, and conduct their analysis;
independently see the consequences of the results;
accurately represent mathematical knowledge in the field of complex computing in oral and written form.

master:

skills of mastering a large amount of information and solving problems (including complex ones);
skills of independent work and mastering new disciplines;
the culture of the formulation, analysis and solution of mathematical and applied problems that require the use of mathematical approaches and discrete geometry methods for their solution;
the subject language of complex calculations and the skills of competent description of problem solving and presentation of the results.

4. Content of the course (training module), structured by topics (sections), indicating the number of allocated academic hours and types of training sessions

4.1. The sections of the course (training module) and the complexity of the types of training sessions

№	Topic (section) of the course	Types of training sessions, including independent work			
		Lectures	Seminars	Laboratory practical	Independent work
1	Examples of combinatorial geometry in the line and in the plane. Basic notions of convex geometry.	6	6		6
2	Extremal points of convex sets, examples. Existence of extremal points for convex compact sets.	6	6		6
3	Faces of polytopes, the face lattice and its behavior under polarity. Basic results of combinatorial geometry in arbitrary dimension.	6	6		6
4	Applications of Helly's theorem:	6	6		6
5	Kirchberger's theorem about separation by hyperplanes. Examples of the minimization technique.	6	6		6
6	Helly-type theorems for algebraic sets. Voronoi partitions and Delaunay triangulations. Correctness of the flipping algorithm	4	4		6
7	Tilings of the space and Voronoi's conjecture on parallelohedra. Volume of a union of balls.	4	4		8
8	The Kneser–Poulsen problem on the intersection or the union of balls. Tarski plank problem. The Kadets theorem on the total sum of inradii of convex bodies covering unit ball	4	4		7
9	The Danzer–Grünbaum theorem on antipodal sets. The incribed ellipsoid of maximal volume in a convex body.	6	6		8

10	The functional Brunn–Minkowski inequality, the logarithmic Brunn–Minkowski inequality, and the ordinary Brunn–Minkowski inequality. Convexity of the set of differentials of a convex function.	6	6		8
11	The ham-sandwich theorem for measures and signed measures. The Szemer edi–Trotter theorem on point and line incidences. Dol'nikov's theorem on intersection by hyperplanes and the chromatic number of the Kneser graph.	6	6		8
AH in total		60	60		75
Exam preparation		30 AH.			
Total complexity		225 AH., credits in total 5			

4.2. Content of the course (training module), structured by topics (sections)

Semester: 1 (Fall)

1. Examples of combinatorial geometry in the line and in the plane. Basic notions of convex geometry.

Covering, coloring, and piercing. The Hahn–Banach separation theorem (finite-dimensional case for closed sets). The polar body and the polar cone.

2. Extremal points of convex sets, examples. Existence of extremal points for convex compact sets.

Dimension of a convex set and its relative interior, stability of the relative interior under taking the closure. Separation from not necessarily closed sets. The Krein–Milman theorem for convex compact sets in Rn. Polytopes and polyhedra.

3. Faces of polytopes, the face lattice and its behavior under polarity. Basic results of combinatorial geometry in arbitrary dimension.

Simple and simplicial polytopes, h-vector of a simple polytope and the Dehn–Sommerville relations. Geometric view on linear programming. Basic results of combinatorial geometry in arbitrary dimension. Carath ́eodory's, Radon's, and Helly's theorems.

4. Applications of Helly's theorem:

Compactness of the convex hull of a compact set, the Jung inequality, the centerpoint theorem, covering a point set by a translate of a convex body.

5. Kirchberger's theorem about separation by hyperplanes. Examples of the minimization technique.

Kirchberger's theorem about separation by hyperplanes. Examples of the minimization technique. The colorful Carath ́eodory theorem and the colorful Helly theorem by ́ar ́any and Lov ́asz. Tverberg's theorem. Basic point-line incidence problems. Sylvester's lemma and number of lines determined by a finite point set. The usage of point-line polarity.

Semester: 2 (Spring)

6. Helly-type theorems for algebraic sets. Voronoi partitions and Delaunay triangulations. Correctness of the flipping algorithm

Helly-type theorems for algebraic sets. Piercing algebraic sets and Helly–Gallai-type theorems. Voronoi partitions and Delaunay triangulations. A Delaunay triangulation in the plane from the algorithmic viewpoint. Existence proof: A Delaunay triangulation as the convex hull of the points lifted to a paraboloid (or to a sphere). A Voronoi partition as a projection of a polyhedron outscribed about the paraboloid (or about the sphere). Correctness of the flipping algorithm: A sequence of flips as a triangulation of this convex hull. Generalized Voronoi partitions (regular partitions).

7. Tilings of the space and Voronoi's conjecture on parallelohedra. Volume of a union of balls.

Tilings of the space and Voronoi's conjecture on parallelohedra. Shellability of polyhedra, ordering properties of Voronoi partitions and Delaunay triangulations. The notion of a simplicial or cellular complex and its Euler characteristic. Volume of a union of balls. Generalized Voronoi partition corresponding to a set of balls. Simplification of the inclusion-exclusion formula for the volume of a union of balls by Edelsbrunner. Kirszbraun's theorem and extension of Lipschitz maps by kopyan. Reduction of Jung's theorem to Kirszbraun's theorem.

8. The Kneser–Poulsen problem on the intersection or the union of balls. Tarski plank problem. The Kadets theorem on the total sum of inradii of convex bodies covering unit ball

The Kneser–Poulsen problem on the intersection or the union of balls. The continuous motion case of the Kneser–Poulsen problem by Gromov and by Csikos. Reduction of the case of at most n + 1 ball in Rn or Sn to the case of continuous motion. Tarski plank problem, Bang's proof of Tarski plank problem. Proof of Ĺaszl ́o Fejes Tothzone conjecture. The Kadets theorem on the total sum of inradii of convex bodies covering unit ball. Kadets-type results by Karasev and Akopyan.

9. The Danzer–Gr ̈unbaum theorem on antipodal sets. The incribed ellipsoid of maximal volume in a convex body.

The Danzer–Grünbaum theorem on antipodal sets. Arrangements of pairwise touching or intersecting translates and homothets. Applications to the problem on maximal cardinality of k-distants sets. The incribed ellipsoid of maximal volume in a convex body. John's theorem. Quantative Helly's theorem by Naszódi. Minkowski's theorem on areas of facets: A proof of the equality. Triangulation of polytopes and their "circumcenter of mass" by Akopyan.

10. The functional Brunn–Minkowski inequality, the logarithmic Brunn–Minkowski inequality, and the ordinary Brunn–Minkowski inequality. Convexity of the set of differentials of a convex function.

The functional Brunn–Minkowski inequality, the logarithmic Brunn–Minkowski inequality, and the ordinary Brunn–Minkowski inequality. Logarithmic concavity and the Prékopa–Leindler inequality. Convexity of the set of differentials of a convex function. Minkowski's theorem: A proof of existence and uniqueness. The Knaster–Kuratowski–Mazurkiewicz theorem and Brouwer's theorem on fixed points. Applications: piersing number of d-intervals. The Borsuk–Ulam theorem for odd maps between spheres. Equivalent versions of the Borsuk–Ulam theorem; the degree of an odd map of a sphere into itself. Crofton's formula and the preimage of zero for an odd map of a sphere into a Euclidean space.

11. The ham-sandwich theorem for measures and signed measures. The Szemerédi–Trotter theorem on point and line incidences. Dol'nikov's theorem on intersection by hyperplanes and the chromatic number of the Kneser graph.

The ham-sandwich theorem for measures and signed measures. The moment curve and its generalizations, the polynomial ham-sandwich theorem. The Szemerédi–Trotter theorem on point and line incidences. Some easy sum and product set estimates for a set of real numbers. Dol'nikov's theorem on intersection by hyperplanes and the chromatic number of the Kneser graph. The canonical vector bundle over the Grassmann manifold, coincidence of its sections, and Dol'nikov's transversal theorem. The central transversal theorem.

5. Description of the material and technical facilities that are necessary for the implementation of the educational process of the course (training module)

A standard classroom.

6. List of the main and additional literature, that is necessary for the course (training module) mastering

Main literature

1. Классическая дифференциальная геометрия [Текст] / А. И. Шафаревич ; М-во образования и науки Рос. Федерации, Моск. физ.-техн. ин-т (гос. ун-т - М.МФТИ,2010

1. Аналитическая геометрия и линейная алгебра [Текст] : учеб. пособие для вузов / А. Е. Умнов ; М-во образования и науки РФ, Моск. физ.-техн. ин-т (гос. ун-т .— 3-е изд., испр. и доп .— М. : МФТИ, 2011 .— 544 с. +pdf. версия. - Библиогр.: с. 528. - Предм. указ.: с. 529-543. - 400 экз. - ISBN 978-5-7417-0378-6 (в пер.) .— Полный текст (Режим доступа : доступ из сети МФТИ).

Additional literature

1. Дискретная математика [Текст] : учеб. пособие для вузов / А. Н. Макоха, П. А Сахнюк, Н. И. Червяков .— М. : Физматлит, 2005 .— 368 с. - Библиогр.: с. 366-368. - ISBN 5-9221-0630-9 (в пер.) .— Полный текст (Доступ из сети МФТИ / Удаленный доступ).

7. List of web resources that are necessary for the course (training module) mastering

http://dm.fizteh.ru/

8. List of information technologies used for implementation of the educational process, including a list of software and information reference systems (if necessary)

Multimedia technologies can be employed during lectures and practical lessons, including presentations.

9. Guidelines for students to master the course

1. It is recommended to successfully pass the test papers, as this simplifies the final certification in the subject.
2. To prepare for the final certification in the subject, it is best to use the lecture materials.

SUPPLEMENT

Assessment funds for course (training module)

major:	Applied Mathematics and Informatics
specialization:	Advanced Methods of Modern Combinatorics/Продвинутые методы современной комбинаторики
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	Chair of Discrete Mathematics
term:	1
qualification:	Master

Semesters, forms of interim assessment:

1 (fall) - Grading test

2 (spring) - Exam

Author: A.A. Polyanskiy, candidate of physics and mathematical sciences, associate professor

1. Competencies formed during the process of studying the course

Code and the name of the competence	Competency indicators
UC-2 Able to manage a project through all stages of its life cycle	UC-2.4 Publicly present the project results (or results of its stages) via reports, articles, presentations at scientific conferences, seminars, and similar events
Gen.Pro.C-1 Address current challenges in fundamental and applied mathematics	Gen.Pro.C-1.1 Apply fundamental scientific knowledge, new scientific principles, and research methods in applied mathematics and computer science
Gen.Pro.C-3 Develop mathematical models and conduct their analysis in the processes of professional problem-solving	Gen.Pro.C-3.2 Employ research methods to solve new problems, and apply knowledge from various science and technology fields
Pro.C-1 Become part of a professional community and conduct local research under scientific guidance using methods specific to a particular professional setting	Pro.C-1.1 Apply principles of scientific work, methods of data collection and analysis, ways of argumentation; prepare scientific reviews, publications, abstracts, and bibliographies on research topics in Russian and English
	Pro.C-1.3 Use practical knowledge of scientific argumentation when analyzing a research subject area
Pro.C-3 Participate in scholarly discussions, make speeches and presentations (oral, written, and online) on scientific topics, present research materials, proofread, edit, reference scientific works	Pro.C-3.1 Learn the basics of scholarly discussion and the forms of verbal scientific communication

2. Competency assessment indicators

As a result of studying the course the student should:

know:

fundamental concepts, laws, theories of discrete geometry;
current problems of the corresponding sections of discrete geometry;
concepts, axioms, methods of proof and proof of the main theorems in the sections included in the basic part of the cycle;
basic properties of the corresponding mathematical objects;
analytical and numerical approaches and methods for solving typical applied problems of discrete geometry.

be able to:

understand the task;
use your knowledge to solve fundamental and applied problems of discrete geometry;
evaluate the correctness of the problem statements;
strictly prove or disprove the statement;
independently find algorithms for solving problems, including non-standard ones, and conduct their analysis;
independently see the consequences of the results;
accurately represent mathematical knowledge in the field of complex computing in oral and written form.

master:

skills of mastering a large amount of information and solving problems (including complex ones);
skills of independent work and mastering new disciplines;
the culture of the formulation, analysis and solution of mathematical and applied problems that require the use of mathematical approaches and discrete geometry methods for their solution;
the subject language of complex calculations and the skills of competent description of problem solving and presentation of the results.

3. List of typical control tasks used to evaluate knowledge and skills

Current control

1. Implement local methods for improving a triangular discrete surface. In what cases can it be reduced to the Delaunay grid?

2. How to distribute points on the sphere so that the volume of the convex hull is maximal?

3.Use the thermal conductivity descriptor to smooth the surface. What surface properties can you extract with it?

4. How many components does the Helmholtz-Hodge expansion of a vector field have? What properties do they have?

5. Define flows (vector fields) of principal curvatures on the discrete surface and enter local coordinates along them.

4. Evaluation criteria

Set-off questions:

1) Is there a complex 2x4-matrix of 2x4-minors which (randomly written) is a) 2,3,4,5,6,7 b) 3,4,5,6,7,8? If yes, give an example of such a matrix; if not, explain why.

2) Show that a field whose additive group is finitely generated is, of course, as a set.

3) Show that a smooth flat quartic has 28 bi-tangents or one 4-fold tangent.

4) Show that classes of proportional mxn-matrices of rank at most k form an irreducible projective variety and find its dimension.

5) We denote by P the projective space of plane cubic curves passing through the given 6 points, none of which 3 lie on the same line, and all 6 do not lie on the conic. The map f associates with each point of a plane other than 6 data a hyperplane in P formed by all cubic curves from P passing through this point. Find dim P, show that the mapping f is well defined and that the closure of its image (in the projective space dual to P) is a smooth cubic surface, and explicitly describe 27 pencil bundles of cubic curves from P that go into 27 lines on this surface.

Exam Ticket Examples

Ticket 1

1. Applications of Helly's theorem.

2. Tilings of the space and Voronoi's conjecture on parallelohedra.

Ticket 2

1. The functional Brunn–Minkowski inequality.

2. Faces of polytopes, the face lattice and its behavior under polarity.

Assessment "excellent (10)" is given to a student who has displayed comprehensive, systematic and deep knowledge of the educational program material, has independently performed all the tasks stipulated by the program, has deeply studied the basic and additional literature recommended by the program, has been actively working in the classroom, and understands the basic scientific concepts on studied discipline, who showed creativity and scientific approach in understanding and presenting educational program material, whose answer is characterized by using rich and adequate terms, and by the consistent and logical presentation of the material;

Assessment "excellent (9)" is given to a student who has displayed comprehensive, systematic knowledge of the educational program material, has independently performed all the tasks provided by the program, has deeply mastered the basic literature and is familiar with the additional literature recommended by the program, has been actively working in the classroom, has shown the systematic nature of knowledge on discipline sufficient for further study, as well as the ability to amplify it on one's own, whose answer is distinguished by the accuracy of the terms used, and the presentation of the material in it is consistent and logical;

Assessment "excellent (8)" is given to a student who has displayed complete knowledge of the educational program material, does not allow significant inaccuracies in his answer, has independently performed all the tasks stipulated by the program, studied the basic literature recommended by the program, worked actively in the classroom, showed systematic character of his knowledge of the discipline, which is sufficient for further study, as well as the ability to amplify it on his own;

Assessment “good (7)” is given to a student who has displayed a sufficiently complete knowledge of the educational program material, does not allow significant inaccuracies in the answer, has independently performed all the tasks provided by the program, studied the basic literature recommended by the program, worked actively in the classroom, showed systematic character of his knowledge of the discipline, which is sufficient for further study, as well as the ability to amplify it on his own;

Assessment “good (6)” is given to a student who has displayed a sufficiently complete knowledge of the educational program material, does not allow significant inaccuracies in his answer, has independently carried out the main tasks stipulated by the program, studied the basic literature recommended by the program, showed systematic character of his knowledge of the discipline, which is sufficient for further study;

Assessment “good (5)” is given to a student who has displayed knowledge of the basic educational program material in the amount necessary for further study and future work in the profession, who while not being sufficiently active in the classroom, has nevertheless independently carried out the main tasks stipulated by the program, mastered the basic literature recommended by the program, made some errors in their implementation and in his answer during the test, but has the necessary knowledge for correcting these errors by himself;

Assessment “satisfactory (4)” is given to a student who has discovered knowledge of the basic educational program material in the amount necessary for further study and future work in the profession, who while not being sufficiently active in the classroom, has nevertheless independently carried out the main tasks stipulated by the program, learned the main literature but allowed some errors in their implementation and in his answer during the test, but has the necessary knowledge for correcting these errors under the guidance of a teacher;

Assessment “satisfactory (3)” is given to a student who has displayed knowledge of the basic educational program material in the amount necessary for further study and future work in the profession, not showed activity in the classroom, independently fulfilled the main tasks envisaged by the program, but allowed errors in their implementation and in the answer during the test, but possessing necessary knowledge for elimination under the guidance of the teacher of the most essential errors;

Assessment “unsatisfactory (2)” is given to a student who showed gaps in knowledge or lack of knowledge on a significant part of the basic educational program material, who has not performed independently the main tasks demanded by the program, made fundamental errors in the fulfillment of the tasks stipulated by the program, who is not able to continue his studies or start professional activities without additional training in the discipline in question;

Assessment “unsatisfactory (1)” is given to a student when there is no answer (refusal to answer), or when the submitted answer does not correspond at all to the essence of the questions contained in the task.

5. Methodological materials defining the procedures for the assessment of knowledge, skills, abilities and/or experience

During examination the student are allowed to use the program of the discipline.