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On hypergraph cliques with chromatic number 3 and a given number of vertices

In 1973, P. Erdős and L. Lovász pointed out that any hypergraph with pairwise intersecting edges has chromatic number 2 or 3. In the first case, this hypergraph can have any number of edges. However, Erdős and Lovász proved that in the second case, the number of edges is bounded from above. For example, if a hypergraph is n-uniform, has pairwise intersecting edges and chromatic number 3, the number of its edges is less than n^n. Recently, D.D. Cherkashin improved this bound (see [2]). In this paper, we further improve it, when the number of vertices of an n-uniform hypergraph is bounded from above by the value n^m with some $m = m(n)$.

Keywords: hypergraph clique, chromatic number.

1. Introduction and the main result

This paper is devoted to a problem in extremal hypergraph theory, which goes back to P. Erdős and L. Lovász (see [3]). Before giving the exact statement of the problem, we recall some definitions and introduce some notation.

Let $H = (V, E)$ be a hypergraph without multiple edges. We call it *n-uniform* if any of its edges has cardinality n: for every $e \in E$, we have $|e| = n$. The *chromatic number* of the hypergraph $H = (V, E)$ is the minimum number $\chi(H)$ of colors needed to color all the vertices in V, so that any edge $e \in E$ contains at least two vertices of some different colors. Finally, a hypergraph is said to form a *clique* if its edges are pairwise intersecting.

In 1973, Erdős and Lovász pointed out that if an n-uniform hypergraph $H = (V, E)$ forms a clique, then $\chi(H) \in \{2, 3\}$. They also observed that in the case $\chi(H) = 3$, one is sure to have $|E| \leqslant n^n$ (see [3]). Thus, the following definition is justified:

$$M(n) = \max\{|E| : \exists \text{ an } n-\text{uniform clique } H = (V, E) \text{ with } \chi(H) = 3\}.$$

Obviously, this definition has no meaning for $\chi(H) = 2$.

Theorem 1 (P. Erdős, L. Lovász, [3]). *The inequalities hold*

$$n!\left(\frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) \leqslant M(n) \leqslant n^n.$$

Almost nothing better is done in the past 35 years. In [5], the estimate $M(n) \leqslant \left(1 - \frac{1}{e}\right) n^n$ is mentioned as “to appear”. However, we fail to find the corresponding paper.

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At the same time, another value $r(n)$ is introduced in [6]:

$$r(n) = \max\{|E| : \ \exists \text{ an } n-\text{uniform clique } H = (V, E) \text{ s.t. } \tau(H) = n\},$$

where $\tau(H)$ is the *covering number* of H, i.e.,

$$\tau(H) = \min\{|f| : \ f \subset V, \ \forall\, e \in E \ \ f \cap e \neq \varnothing\}.$$

Clearly, for any n-uniform clique H, we have $\tau(H) \leqslant n$ (since every edge forms a cover), and if $\chi(H) = 3$, then $\tau(H) = n$. Hence, $M(n) \leqslant r(n)$. Lovász pointed out that, for $r(n)$, the same estimates, as in Theorem 1, apply and conjectured that the lower estimate is best possible. In 1996, P. Frankl, K. Ota, and N. Tokushige (see [4]) disproved this conjecture and showed that $r(n) \geqslant \left(\frac{n}{2}\right)^{n-1}$.

In [2], D.D. Cherkashin discovered a new upper bound on the initial value of $M(n)$, which is actually true for $r(n)$ as well.

Theorem 2 (D.D. Cherkashin, [2]). *There exists a constant $c > 0$ such that*

$$M(n) \leqslant c n^{n-\frac{1}{2}} \ln n.$$

To present the main result of this paper we take any natural numbers n, $m \geqslant 2$, and put $q(n, m) = \left[\frac{n}{2m}\right]$,

$$A(n, m) = \sum_{i=0}^{2q(n,m)} \binom{n^m}{i}.$$

We note that

$$A(n, m) \leqslant \left(\frac{n}{m} + 1\right) \binom{n^m}{2q(n, m)} \leqslant \left(\frac{n}{m} + 1\right) \left(\frac{e n^m}{2q(n, m)}\right)^{n/m} = n^n \cdot A'(n, m),$$

where

$$A'(n, m) = \left(\frac{n}{m} + 1\right) \left(\frac{e}{2q(n, m)}\right)^{n/m}.$$

Obviously, if m is a function of n, which is $o(n)$ as $n \to \infty$, then

$$A'(n, m) = \frac{m}{n \omega(n)},$$

where $\omega(n) \to \infty$ as $n \to \infty$. Hence, $A(n, m) = o\left(m n^{n-1}\right)$.

Theorem 3. *Let $m \geqslant 2$ be any function of $n \in \mathbb{N}$ which is $o(n)$ as $n \to \infty$; moreover, $m(n) \leqslant \frac{n}{2}$. For any $n \geqslant 4$ and any n-uniform clique $H = (V, E)$ with $\chi(H) = 3$ and $|V| \leqslant n^{m(n)}$, we have*

$$|E| \leqslant 4m(n)n^{n-1} + A(n, m(n)) = (4 + o(1))m(n)n^{n-1}.$$

Clearly, if $m(n) \leqslant c\sqrt{n} \ln n$ with some constant $c > 0$, the bound in Theorem 3 is stronger than that in Theorem 2. Note that the number of vertices in any n-uniform clique with chromatic number 3 is less than 4^n (see [3]). Unfortunately, $n^{\sqrt{n} \ln n} = e^{o(n)}$, so that Theorem 3 does not cover all possible values of $|V|$.

2. Proof of Theorem 3

Let us fix $n \geqslant 4$ and put $m = m(n)$, $q = q(n, m)$, $A = A(n, m)$. Fix an n-uniform clique $H = (V, E)$ with $\chi(H) = 3$ and $|V| \leqslant n^m$. For any set $W \subseteq V$, we denote by $E(W)$ the set of all edges $B \in E$ such that $W \subseteq B$. Also, we denote by E_W the set of all edges $B \in E$ such that $W \cap B \neq \varnothing$. Clearly, $E(W) \subseteq E_W$. Let

$$Q = \{1, 2, 3, \ldots, q\} \cup \{n - q + 1, n - q + 2, \ldots, n\}.$$

The two parts of the set Q do not intersect and do not cover the whole set $\{1, \ldots, n\}$, for $m \geqslant 2$. Moreover, Q is not empty, for $m \leqslant \frac{n}{2}$, hence $q \geqslant 1$.

Lemma 1. *Let $W \subseteq V$, $i = |W|$. Either there exists a vertex $x \in W$ such that* $\deg x \geqslant \dfrac{|E| - A}{i}$, *or there exist two edges $B_1, B_2 \in E$ such that $B_1, B_2 \notin E_W$ and $|B_1 \cap B_2| \notin Q$.*

Proof of Lemma 1. If there exists a vertex $x \in W$ such that $\deg x \geqslant \dfrac{|E| - A}{i}$, we are done. If there are no such vertices, then

$$|E_W| \leqslant \sum_{x \in W} \deg x < |E| - A.$$

Hence, $|E \setminus E_W| > A$. We are to show that there exist two edges $B_1, B_2 \in E \setminus E_W$ with $|B_1 \cap B_2| \notin Q$. Suppose to the contrary that for any $B_1, B_2 \in E \setminus E_W$, we have $|B_1 \cap B_2| \in Q$. We further prove that $|E \setminus E_W| \leqslant A$. We find a contradiction and thus complete the proof of Lemma 1.

In principle, we can only cite [8]. Instead, we use a version of the linear algebra method in combinatorics (see [1] and [7]). To any edge B from $E \setminus E_W$ we assign the vector $\mathbf{x} = (x_1, \ldots, x_v) \in \{0, 1\}^v$, where $v = |V| \leqslant n^m$ and $x_\nu = 1$, if and only if $\nu \in B$. In particular, $x_1 + \ldots + x_v = n$. Let $E \setminus E_W \to \{\mathbf{x}_1, \ldots, \mathbf{x}_s\}$.

We denote by $(\mathbf{x}, \mathbf{y})$ the Euclidean inner product of vectors $\mathbf{x}, \mathbf{y}$. Note that if $B, B' \in E \setminus E_W$ and $\mathbf{x}, \mathbf{x}'$ are the corresponding vectors, then $|B \cap B'| = (\mathbf{x}, \mathbf{x}')$.

We take an arbitrary vector $\mathbf{x}_\nu$, $\nu \in \{1, \ldots, s\}$, and consider the polynomial

$$F_{\mathbf{x}_\nu}(\mathbf{y}) = \prod_{j \in Q \setminus \{n\}} (j - (\mathbf{x}_\nu, \mathbf{y})) \in \mathbb{R}[y_1, \ldots, y_v].$$

Finally, we get s polynomials $F_{\mathbf{x}_1}, \ldots, F_{\mathbf{x}_s}$. All of them depend on v variables and their degree is less than $|Q| \leqslant 2q$. Certainly, any such polynomial is a linear combination of some monomials of the type

$$1, \quad y_{\nu_1}^{\alpha_{\nu_1}} \cdot \ldots \cdot y_{\nu_r}^{\alpha_{\nu_r}}, \quad \alpha_{\nu_1}, \ldots, \alpha_{\nu_r} \geqslant 1, \quad \alpha_{\nu_1} + \ldots + \alpha_{\nu_r} \leqslant |Q| \leqslant 2q.$$

We replace each monomial of this type by $y_{\nu_1} \cdot \ldots \cdot y_{\nu_r}$ and denote by $F'_{\mathbf{x}_1}, \ldots, F'_{\mathbf{x}_s}$ the resulting polynomials. They also depend on v variables and their degree is less than $|Q| \leqslant 2q$. Moreover, they span a linear space whose dimension is less than or equal to

$$\sum_{r=0}^{2q} \binom{v}{r} \leqslant \sum_{r=0}^{2q} \binom{n^m}{r} = A.$$

Simultaneously, $F'_{\mathbf{x}_\nu}(\mathbf{y}) = F_{\mathbf{x}_\nu}(\mathbf{y})$, provided that $\mathbf{y} \in \{0, 1\}^v$ and $\nu \in \{1, \ldots, s\}$.

To show that $s = |E \setminus E_W| \leqslant A$ (needed to complete the proof) it suffices to establish the linear independence of the polynomials $F'_{\mathbf{x}_1}, \ldots, F'_{\mathbf{x}_s}$ over $\mathbb{R}$. Suppose that

$$c_1 F'_{\mathbf{x}_1}(\mathbf{y}) + \ldots + c_s F'_{\mathbf{x}_s}(\mathbf{y}) = 0.$$

Let $\mathbf{y} = \mathbf{x}_\nu$, $\nu \in \{1, \ldots, s\}$. Then $(\mathbf{x}_\nu, \mathbf{y}) = (\mathbf{x}_\nu, \mathbf{x}_\nu) = n$ and

$$F'_{\mathbf{x}_\nu}(\mathbf{y}) = F_{\mathbf{x}_\nu}(\mathbf{y}) = F_{\mathbf{x}_\nu}(\mathbf{x}_\nu) \neq 0.$$

However, if $\mu \neq \nu$, then $(\mathbf{x}_\mu, \mathbf{y}) = (\mathbf{x}_\mu, \mathbf{x}_\nu) \in Q \setminus \{n\}$, i. e.

$$F'_{\mathbf{x}_\mu}(\mathbf{y}) = F_{\mathbf{x}_\mu}(\mathbf{y}) = F_{\mathbf{x}_\mu}(\mathbf{x}_\nu) = 0.$$

Hence, $c_\nu = 0$ for every ν. Lemma 1 is proved.

Lemma 2. *Let $W \subseteq V$, $i = |W|$, $j = |E(W)|$. Suppose that there exist two edges $B_1, B_2 \in \in E \setminus E_W$ such that $|B_1 \cap B_2| \notin Q$. We put $\tau = 1 + \frac{1}{4m}$. Either there exists $x \notin W$ such that $|E(W \cup \{x\})| \geqslant \frac{j\tau}{n}$, or there exist $x, y \notin W$ such that $|E(W \cup \{x, y\})| \geqslant \frac{j\tau^2}{n^2}$.*

Proof of Lemma 2. Let $l = |B_1 \cap B_2| \notin Q$. Consider the set $E(W)$. Since H is a clique, any edge $B \in E(W)$ intersects both B_1 and B_2. Either B intersects the set $B_1 \cap B_2$, or it has common vertices with both $B_1 \setminus (B_1 \cap B_2)$ and $B_2 \setminus (B_1 \cap B_2)$. We denote by E_1 the set of edges of the first type; $E_2 = E(W) \setminus E_1$. By pigeon-hole principle, there is $x \in B_1 \cap B_2$ such that x belongs to at least $\frac{|E_1|}{l}$ edges from E_1; there are also $x \in B_1 \setminus (B_1 \cap B_2)$ and $y \in B_2 \setminus (B_1 \cap B_2)$ such that the set $\{x, y\}$ belongs to at least $\frac{|E_2|}{(n-l)^2}$ edges from E_2. We are to show that for any partition $E(W) = E_1 \cup E_2$, we have

$$\text{either} \quad \frac{|E_1|}{l} \geqslant \frac{j\tau}{n}, \quad \text{or} \quad \frac{|E_2|}{(n-l)^2} \geqslant \frac{j\tau^2}{n^2},$$

which is equivalent to

$$\max\left\{\frac{|E_1|^2}{j^2 l^2}, \frac{|E_2|}{j(n-l)^2}\right\} \geqslant \frac{\tau^2}{n^2}.$$

Here the worst case is that of $\frac{|E_1|^2}{j^2 l^2} = \frac{|E_2|}{j(n-l)^2}$. Let $a = |E_1|$. Then $|E_2| = j - a$ and we have $\frac{a^2}{j^2 l^2} = \frac{j-a}{j(n-l)^2}$. Solving this equation, we get

$$a = \frac{jl^2 + \sqrt{(jl^2)^2 + 4j^2 l^2 (n-l)^2}}{2(n-l)^2}.$$

Of course, the value of $|E_1|$ (which is an integer) can differ from the real number a. However, we do know that

$$\max\left\{\frac{|E_1|^2}{j^2 l^2}, \frac{|E_2|}{j(n-l)^2}\right\} \geqslant \frac{a^2}{j^2 l^2}.$$

Thus, we are to prove that $\frac{a}{jl} \geqslant \frac{\tau}{n}$, or $\frac{an}{jl} \geqslant \tau$. We have

$$\frac{an}{jl} = \frac{l + \sqrt{l^2 + 4(n-l)^2}}{2(n-l)^2} n = \frac{ln}{2(n-l)^2} + \sqrt{\left(\frac{ln}{2(n-l)^2}\right)^2 + \frac{n^2}{(n-l)^2}} \geqslant 1 + \frac{ln}{2(n-l)^2}.$$

The function $\frac{ln}{2(n-l)^2}$ is monotone increasing at l. Since $l \notin Q$, we can use the bound $l \geqslant \frac{n}{2m}$. Consequently,

$$\frac{an}{jl} \geqslant 1 + \frac{ln}{2(n-l)^2} \geqslant 1 + \frac{n^2}{4m(n-l)^2} \geqslant 1 + \frac{1}{4m} = \tau.$$

Lemma 2 is proved.

Completion of the proof of Theorem 3. Let

$$k = \min\left\{|W| : \ W \subseteq V, \ \exists\, x \in W \ \ \deg x \geqslant \frac{|E| - A}{|W|}\right\}.$$

The value k is well-defined. Indeed, we take any edge $W \in E$. Since H is a clique, W intersects all the edges from E and so there exists $x \in W$ with $\deg x \geqslant \frac{|E|}{|W|} > \frac{|E| - A}{|W|}$.

Let W_0 be a set on which the value of k is reached. We take a vertex $x \in W_0$ that has $\deg x \geqslant \frac{|E|-A}{k}$. The latter inequality can be rewritten as $|E(\{x\})| \geqslant \frac{|E|-A}{k}$. If $k \geqslant 2$, we can apply Lemmas 1 and 2 to $W = \{x\}$. Thus, we obtain either some set W' of two elements with $|E(W')| \geqslant \frac{|E|-A}{k} \cdot \frac{\tau}{n}$ or some set W'' of three elements with $|E(W'')| \geqslant \frac{|E|-A}{k} \cdot \frac{\tau^2}{n^2}$. We continue this process until we get some set W with $|W| = k$ and $|E(W)| \geqslant \frac{|E|-A}{k} \cdot \frac{\tau^{k-1}}{n^{k-1}}$ (even if $k = 1$, we do have such a set).

In [3], Erdős and Lovász prove that for any n-uniform clique, $H = (V, E)$ with chromatic number 3, if $W \subseteq V$ is of cardinality k, then $|E(W)| \leqslant n^{n-k}$. In our case, we have $\frac{|E|-A}{k} \cdot \frac{\tau^{k-1}}{n^{k-1}} \leqslant n^{n-k}$. Therefore,

$$|E| \leqslant k \cdot n^{n-k} \cdot \frac{n^{k-1}}{\tau^{k-1}} + A = k\frac{n^{n-1}}{\tau^{k-1}} + A.$$

To complete the proof of Theorem 3 we are to show that for any k, $\frac{k}{\tau^{k-1}} \leqslant 4m$. It is easy to see that the maximum value of $\frac{k}{\tau^{k-1}}$ is obtained for $k = 4m$.

3. Refinement of Theorem 3

For $m = 2$, one can prove a simple result that is, however, much better than that of Theorem 3.

Theorem 4. *Let $H = (V, E)$ be any n-uniform clique with $\chi(H) = 3$. We put $v = |V|$. Suppose that $v \leqslant \frac{n^2}{c}$, where c can be any function of n such that $c(n) \in (1, n)$. We now put*

$$d = ce^{\frac{2}{ec} - 1}.$$

Then

$$|E| \leqslant (1 + o(1))\frac{e^{3/2}}{\sqrt{c}}(n/d)^n.$$

If c is a constant, we get an exponential improvement for the Erdős and Lovász bound, which is equal to n^n. Otherwise, the improvement is even much larger.

Proof of Theorem 4. We take an arbitrary integer $a \in (1, n)$ and consider all a-element subsets of V. The number of such subsets is $\binom{v}{a}$. On the one hand, any edge from E contains exactly $\binom{n}{a}$ subsets. On the other hand, any subset is enclosed in n^{n-a} edges at most (see [3]). Therefore, the number of edges does not exceed the quantity $\frac{n^{n-a}\binom{v}{a}}{\binom{n}{a}}$. To estimate this quantity we use the bound $\binom{v}{a} \leqslant \frac{v^a}{a!}$ and the Stirling formula. Hence,

$$\frac{n^{n-a}\binom{v}{a}}{\binom{n}{a}} \leqslant \frac{n^{n-a}v^a}{a!\binom{n}{a}} \leqslant \frac{n^{n+a}}{c^a \frac{n!}{(n-a)!}}.$$

We now put

$$a = \left[\left(1 - \frac{1}{ec}\right)n\right] + 1.$$

Then $n - a \leqslant \frac{n}{ec}$, so that

$$\frac{n!}{(n-a)!} \sim \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi(n-a)}\left(\frac{n-a}{e}\right)^{n-a}} \geqslant \sqrt{ec} \cdot n^a (e^2 c)^{n-a} e^{-n}$$

and

$$|E| \leqslant (1 + o(1)) \frac{n^{n+a}}{c^a \sqrt{ec} \cdot n^a (e^2 c)^{n-a} e^{-n}} = (1 + o(1)) \frac{n^n}{\sqrt{ec} \cdot c^n e^{-n} e^{2n-2a}} \leqslant$$

$$\leqslant (1 + o(1)) \frac{n^n}{\sqrt{ec} \cdot c^n e^{-n} e^{\frac{2n}{ec} - 2}} = (1 + o(1)) \frac{e^{3/2}}{\sqrt{c}} (n/d)^n.$$

Theorem 4 is proved.

Note that for constant values of c, the choice of a in the proof is nearly optimal.

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