Inequalities of Nonlinearly Tansformed X-states

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THE AIM OF WORK

The aim of our work is to study information properties of X-state under the action of specific nonlinear map. We evaluate the q-entropies for X-states of two qubits and qudit with $j = 3/2$. We consider the specific nonlinear transformation and discuss action of that channel on the entanglement properties. We study the transformation of the q-entropies due to the action of nonlinear channel on the X-states of the four level atom. We also discuss the new entropic subadditivity condition for q-entropy as the quantum correlation property in the system of two qubits and qudit with $j = 3/2$.

INTRODUCTION

The possibility to consider the positive maps of density matrices which are nonlinear maps [1] exists. Here we generalize nonlinear channel considered in [2].
The quantum correlation in two-qubit system have analogs in the noncomposite systems [3]. The analogous to two-qibit case correlations in the single qudit ($j = 3/2$) state are related to dependence of behaviours of the spin projections $m = 3/2, 1/2, -1/2, -3/2$, which shows the dependence of the probability for projection.
We extend the investigation of similarity and difference in statistical properties of composite (two qubits) and noncomposite (qudit $j = 3/2$) systems focusing of the nonlinearly transformed density matrices of these systems.

NONLINEAR CHANNEL OF X-STATE

There are two types of sytems in our consideration. First of them is system of two qubits and another one is system of qudit with $j = 3/2$. The important point is that both types of systems can be described by the density matrices 4×4. Thus, all the mathematical calculation are the same for both cases.
The indexes of matrix rows and colomns in case of two spins can be taken to correspond to states $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$ respectively. For qudit $j = 3/2$ indexes may correspond to projection of angular momentum.
We consider the specific case of states called X-states. They have specific density matrices ρ_X

$$\rho_X = \begin{pmatrix} a & 0 & 0 & f \\ 0 & b & e & 0 \\ 0 & e^* & c & 0 \\ f^* & 0 & 0 & d \end{pmatrix}. \qquad (1)$$

The matrix ρ_X depends on six parameters: a, b, c, d are real numbers and e, f are complex. Due to the property of density matrix $\mathrm{Tr}\rho = 1$ only five parameters are independent. The considered matrix is density matrix in domain ($a \geq 0, b \geq 0, c \geq 0, d \geq 0, ad \geq |f|^2, bc \geq |e|^2$). According to Peres-Horodecki criterion of entanglement [4,5] the state is separable if the positive partial transposed matrix ρ^{ppt} (ppt-matrix) of initial density matrix is nonnegative. The ppt-matrix for X-state is

$$\rho_X^{ppt} = \begin{pmatrix} a & 0 & 0 & e \\ 0 & b & f & 0 \\ 0 & f^* & c & 0 \\ e^* & 0 & 0 & d \end{pmatrix}. \qquad (2)$$

Thus, the state described by ρ_X is separable if $ad \geq |e|^2, |f|^2$ and $bc \geq |e|^2, |f|^2$.
We consider the nonlinear channel given by formula

$$\tilde{\Phi}_t(\rho) = \rho_t = \frac{1}{\mathrm{Tr} D^t} S \cdot D^t \cdot S^{-1}, \qquad (3)$$

where the parameter t of channel is arbitrary real number. The matrix D is diagonal matrix of eigenvalues of ρ_X, and matrix S consists of corresponding eigenvectors of ρ_X taken as colomns. The matrix S is chosen to be unitary. If we made the notations $\xi = \sqrt{(a-d)^2 + 4|f|^2}$, $\zeta = \sqrt{(b-c)^2 + 4|e|^2}$, the elements of matrix S are

$$S_{11} = -S_{44} = \frac{\sqrt{2}|f|}{\sqrt{\xi\,(\xi - (a-d))}},$$
$$S_{22} = -S_{33} = \frac{\sqrt{2}|e|}{\sqrt{\zeta\,(\zeta - (b-c))}},$$
$$S_{14} = S_{41}^* = \frac{\sqrt{2}}{2}\frac{f}{|f|}\sqrt{\frac{\xi - (a-d)}{\xi}},$$
$$S_{23} = S_{32}^* = \frac{\sqrt{2}}{2}\frac{e}{|e|}\sqrt{\frac{\zeta - (b-c)}{\zeta}},$$

and the others matrix elements are zeros.

SEPARABILITY AND ENTANGLEMENT OF TRANSFORMED STATE

The entanglement property is discussed for composite systems (two qubits) and noncomposite systems (qudit $j = 3/2$). The consideration of entanglement properties of composite and noncomposite systems is based on similarity of the problem in terms of density matrices. The density matrix of X-state transformed by nonlinear channel $\tilde{\Phi}_t(\rho_X)$ have the form of some X-state density matrix

$$\rho_{t,X} = \begin{pmatrix} A_t & 0 & 0 & F_t \\ 0 & B_t & E_t & 0 \\ 0 & E_t^* & C_t & 0 \\ F_t^* & 0 & 0 & D_t \end{pmatrix} \qquad (4)$$

and is separable if the conditions

$$A_t D_t \geq |E_t|^2, |F_t|^2$$
$$B_t C_t \geq |E_t|^2, |F_t|^2 \qquad (5)$$

are realized.

SEPARABILITY AND ENTANGLEMENT OF TRANSFORMED STATE

The numbers $A_t, B_t, C_t, D_t, E_t, F_t$ are given by formulas

$$A_t = \frac{1}{\sum_{k=1}^4 \lambda_k^t}\left(\frac{2|f|^2}{\xi\,(\xi - (a-d))}\lambda_1^t + \frac{\xi - (a-d)}{2\xi}\lambda_4^t\right),$$
$$B_t = \frac{1}{\sum_{k=1}^4 \lambda_k^t}\left(\frac{2|e|^2}{\zeta\,(\zeta - (b-c))}\lambda_2^t + \frac{\zeta - (b-c)}{2\zeta}\lambda_3^t\right),$$
$$C_t = \frac{1}{\sum_{k=1}^4 \lambda_k^t}\left(\frac{2|e|^2}{\zeta\,(\zeta - (b-c))}\lambda_3^t + \frac{\zeta - (b-c)}{2\zeta}\lambda_2^t\right), \qquad (6)$$
$$D_t = \frac{1}{\sum_{k=1}^4 \lambda_k^t}\left(\frac{2|f|^2}{\xi\,(\xi - (a-d))}\lambda_4^t + \frac{\xi - (a-d)}{2\xi}\lambda_1^t\right),$$
$$E_t = \frac{e}{\zeta}(\lambda_2^t - \lambda_3^t),\ F_t = \frac{f}{\xi}(\lambda_1^t - \lambda_4^t),$$

where $\lambda_k, k = 1, 2, 3, 4$ are eigenvalues of initial X-state

$$\lambda_{1,4} = \frac{1}{2}(a + d \pm \xi)\ \lambda_{2,3} = \frac{1}{2}(b + c \pm \zeta) \qquad (7)$$

The considered nonlinear channel can change the entanglement of the initial system.

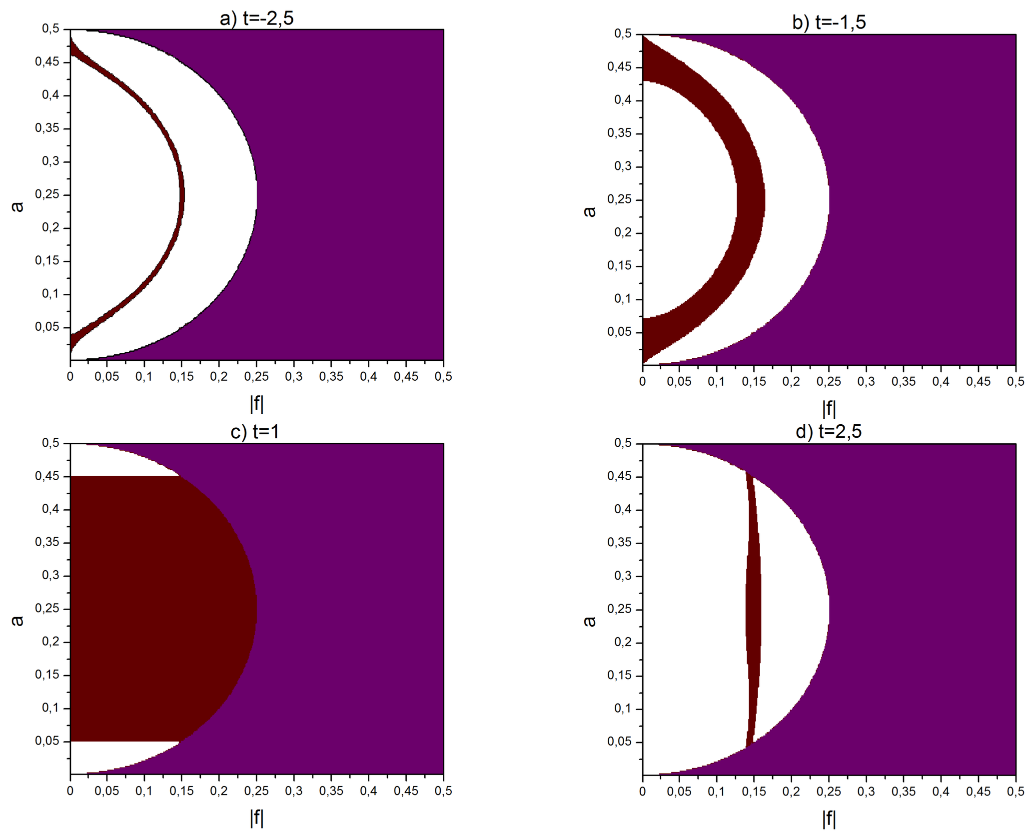

Figure : Domains of parameters a and $|f|$, where the state is separable (white colour), entangled (brown colour). The purple colour corresponds to cases when matrix ρ_X is not the density matrix. The parameters $a + d = 0.5$, $b = c = 0.25$, $|e| = 0.15$. The figures correspond to states transformed by channel with parameter a) $t = -2.5$, b) $t = -1.5$, c) $t = 1$, d) $t = 2.5$.

NEGATIVITY AND CONCURRENCE

To mesure the entanglement of transformed states we use negativity [6] and concurrence [7,8]. Negativity is defined as follows

$$N = \mathrm{Tr}|\rho^{ppt}|, \qquad (8)$$

where ρ^{ppt} - ppt-transformed matrix ρ. $N = 1$ for separable states and $N > 1$ for entangled.
Concurrence can be defined as follows

$$C = max(0, \sqrt{\tilde{\lambda}_1} - \sqrt{\tilde{\lambda}_2} - \sqrt{\tilde{\lambda}_3} - \sqrt{\tilde{\lambda}_4}), \qquad (9)$$

where $\tilde{\lambda}_k, k = 1, 2, 3, 4$ - are eigenvalues of matrix $R = \rho\rho_C$, and $\tilde{\lambda}_1$ has the maximum value. $\rho_C = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, where σ_y is Pauli matrix.
Concurrence takes value 0 for separable states and greater for entangled states.
If $\tilde{\lambda}_1 = \left(\sqrt{A_t D_t} + |F_t|\right)^2$, then the concurrence is

$$C = max(0, 4\sqrt{A_t D_t}|F_t| - 2(|E_t|^2 + B_t C_t)). \qquad (10)$$

The graphs of concurrence are very similar to the corresponding graphs of negativity.

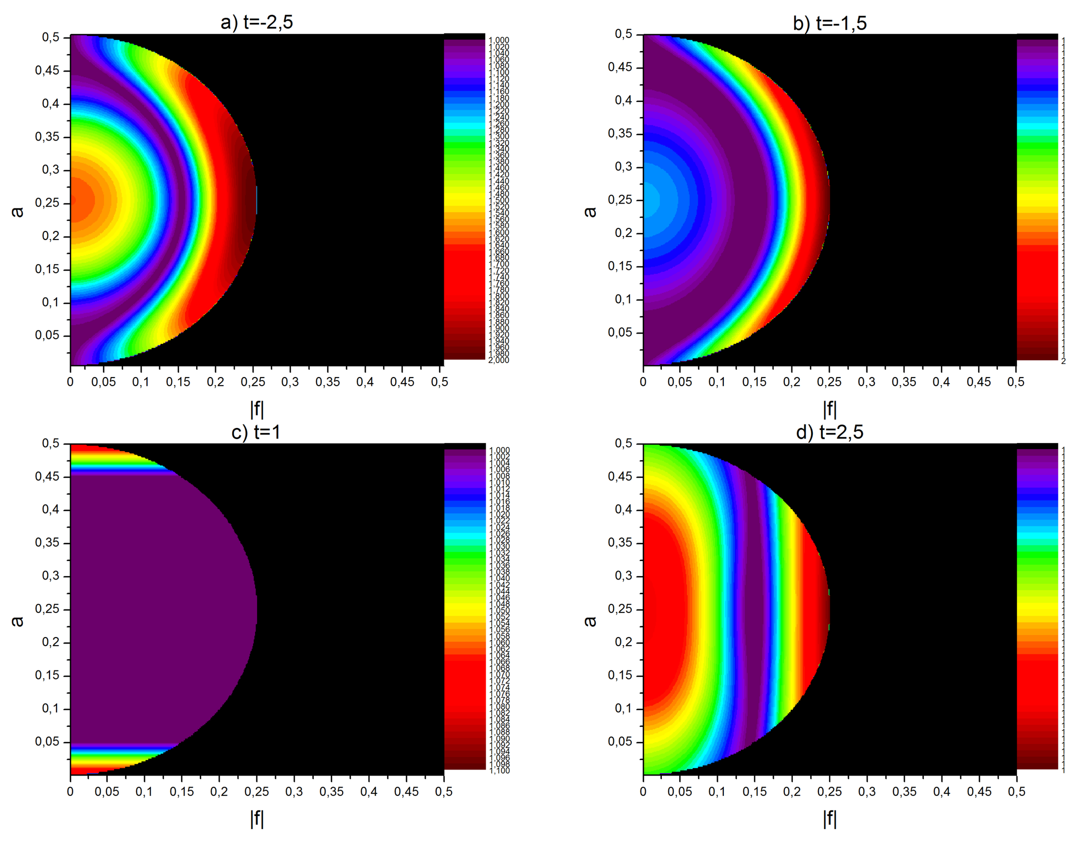

Figure : Negativity for transformed states. The black colour corresponds to cases when matrix ρ_X is negative. The parameters $a + d = 0.5$, $b = c$, $e = 0.15$ for nonlinear channel with a)$n = -2.5$, b)$n = -1.5$, c)$n = 1$ and d)$n = 2.5$.

TSALLIS INFORMATION FOR X-STATE

Introduce the function $\ln_q \rho = \frac{\rho^{q-1} - 1}{q-1}$. Tsallis entropy [9] with parameter q (q-entropy) is given by formula

$$S_q(\rho) = -\mathrm{Tr}\left[\rho \ln_q \rho\right],\ = \frac{1 - \left(\lambda_1^q + \lambda_2^q + \lambda_3^q + \lambda_4^q\right)}{q-1}, \qquad (11)$$

where $\lambda_k, k = 1, 4$ are eigenvalues of ρ. Neumann entropy for state is defined as follows

$$S_N(\rho) = -\mathrm{Tr}[\rho \ln \rho]. \qquad (12)$$

Neumann entropy is particular case of q-entropy at the limit $q = 1$.
For composite system with subsystems (1) and (2) Tsallis information can be introduced as follows

$$I_q(\rho) = S_q(1) + S_q(2) - S_q(1, 2). \qquad (13)$$

For X-state of two-qubit system density matrices $\rho(1)$, $\rho(2)$ of subsystems are obtained by taking partial traces

$$\rho(1) = \begin{pmatrix} a+b & 0 \\ 0 & c+d \end{pmatrix}\ \rho(2) = \begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix}. \qquad (14)$$

The quantum information for qudit $j = 3/2$ can be represented identically in mathematical notations.
For considered systems the subadditivity condition is

$$I_q(\rho_X) = S_q(1) + S_q(2) - S_q(1, 2) \geq 0. \qquad (15)$$

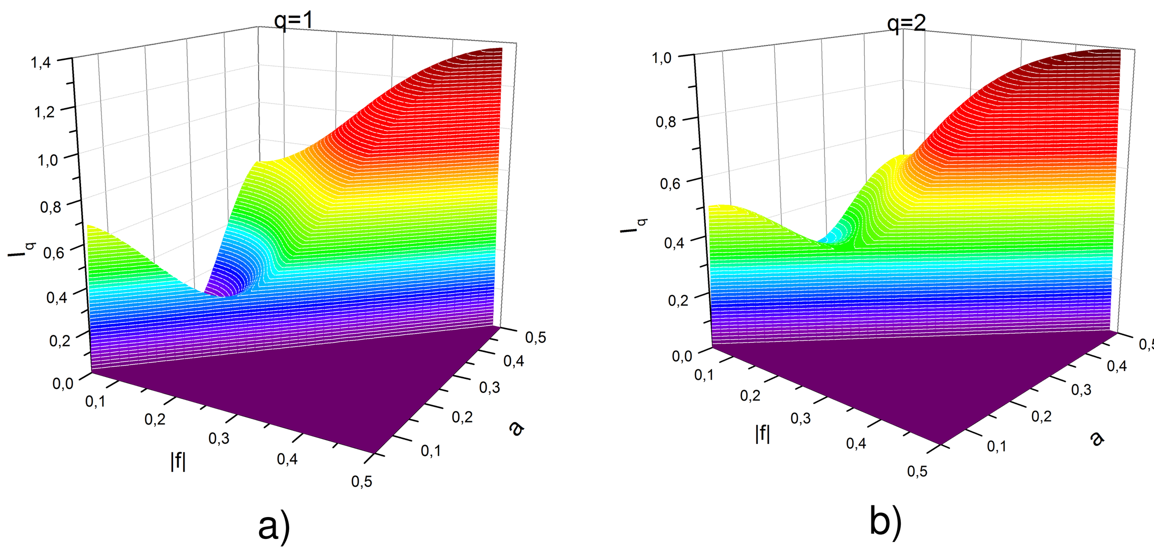

Figure : Tsallis information for transformed X-state $\tilde{\Phi}_t(\rho_X)$ with parameters $a = d$, $b = c$, $c = 0.01$ for a)$q = 1$ and b)$q = 2$.

TSALLIS ENTROPY AND INFORMATION FOR TRANSFORMED X-STATE

Tsallis entropy for transformed states is expressed in terms of $\lambda_k, k = 1, 4$ as follows

$$S_q(\tilde{\Phi}_t) = \frac{1 - \sum_{k=1}^4 \lambda_k^{tq} / \left(\sum_{k=1}^4 \lambda_k^t\right)^q}{q-1}. \qquad (16)$$

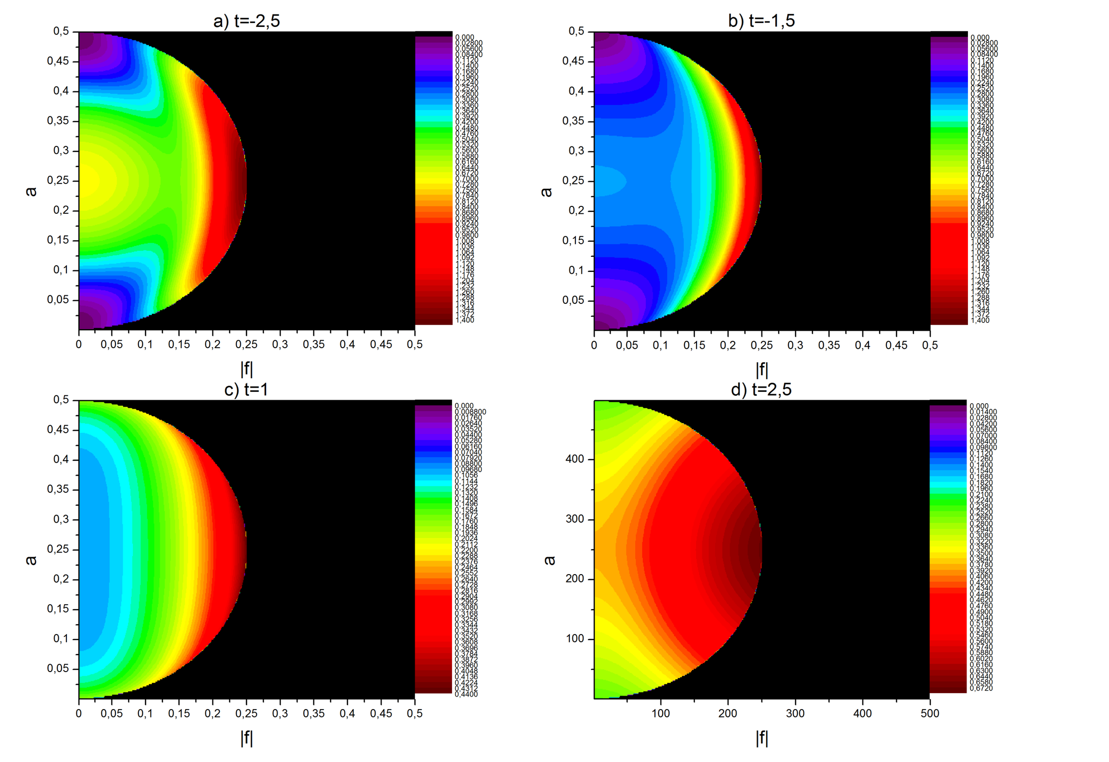

Figure : Tsallis information for X-state with parameters $a + d = 0.5$, $b = c$, $e = 0.15$ for nonlinear map with a)$n = -2.5$, b)$n = -1.5$, c)$n = 1$, d)$n = 2.5$.

All the results for Tsallis entropy and information of X-state are correct for Tsallis entropy and information of transformed state due to the form of matrix $\rho_{t,X}$. Thus, the inequality

$$S_q(1) + S_q(2) - S_q(1, 2) \geq 0 \qquad (17)$$

is satified for Tsallis entropies of transformed X-state.

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CONCLUSION

For X-state of single qudit with $j = 3/2$ and two qubits we constructed the nonlinear positive map depending on real parameter t. The phenomenon of changing of the entanglement properties of the X-state under action of the nonlinear map was observed. The characteristics of the entanglement in terms of concurrence and negativity were evaluated explicitly. The notion of the entanglement known for two-qubit system and characterizing the quantum correlations between the subsystem degrees of freedom was extended to the single qudit state with $j = 3/2$.