Variant 1

1.④ If x is a limit point of X, then every neighbourhood of x has infinitely many points from X, hence it has infinitely many points of Y.

2.⑦ First of all it is easy to compute $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. Then $\lim \frac{e^{2x}-1}{x} = 2$ implies $|f(x,y) - f(0,0)| \leq c|y|^{5/3}$ for sufficiently small $\sqrt{x^2+y^2}$.

3.⑦

$$df(1,0) = \frac{5}{2}dx + \frac{1}{2}dy,\ d^2f(1,0) = \frac{5}{4}dx^2 - \frac{3}{2}dxdy - \frac{3}{4}dy^2,$$

$$f(x,y) = 1 + \frac{5}{2}(x-1) + \frac{1}{2}y + \frac{5}{8}(x-1)^2 - \frac{3}{4}(x-1)y - \frac{3}{8}y^2 + o\left((x-1)^2 + y^2\right).$$

4.⑥ $V = \pi \int_0^1 \arcsin^2 x dx = \pi\, x \arcsin^2 x\Big|_0^1 - 2\pi \int_0^1 x \arcsin x \frac{dx}{\sqrt{1-x^2}} = \frac{\pi^3}{4} + 2\pi \left(\arcsin x\sqrt{1-x^2} - x\right)\Big|_0^1 = \frac{\pi^3}{4} - 2\pi.$

5.⑤

$$f'(x) = \frac{1}{\sqrt{3+x^2}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{x^{2n}}{3^{n+1/2}},$$

$$f(x) = \frac{1}{2}\ln 3 + \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{x^{2n+1}}{3^{n+1/2}(2n+1)}, \quad R = \sqrt{3}.$$

6.④

$$(dz - dy)\tan(z-y) + \frac{(z-y)(dz-dy)}{\cos^2(z-y)} = dx,$$

hence $\mathrm{grad} z = (1/(1+\pi), 1)$.

7.⑥ $(0,0), (1,-1)$ - critical points,

$$d^2 f = (12 - 12x)dx^2 + 12dxdy + 6dy^2,$$

$(0,0)$ is the minimum point, $f(0,0) = 0$, no extrema at $(1,-1)$.

Variant 2

1. ④ $X = \mathbb{Q}^n$, $Y = \mathbb{R}^n$.

2. ⑦ First of all it is easy to compute $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. Then $\lim \frac{\tan 3y}{y} = 3$ implies $|f(x,y) - f(0,0)| \leq c|x|^{3/2}$ for sufficiently small $\sqrt{x^2+y^2}$.

3. ⑦

$$df(0,-1) = -\frac{1}{2}dx + \frac{5}{2}dy, \quad d^2f(0,-1) = 2dy^2,$$

$$f(x,y) = -\frac{1}{2}x + \frac{5}{2}(y+1) + (y+1)^2 + o\left(x^2 + (y+1)^2\right).$$

4. ⑥

$$L = \int_{\sqrt{3}}^{\sqrt{8}} \sqrt{1 + \frac{1}{x^2}}dx = \int_{\sqrt{3}}^{\sqrt{8}} \frac{x^2+1}{x\sqrt{x^2+1}}dx = \sqrt{x^2+1}\Big|_{\sqrt{3}}^{\sqrt{8}} + \int_{\sqrt{3}}^{\sqrt{8}} \frac{dx/x^2}{\sqrt{1+\frac{1}{x^2}}} = 1 + \frac{1}{2}\ln\frac{3}{2}.$$

5. ⑥

$$f'(x) = -\frac{2}{4+x^2} = \sum_{n=0}^{\infty} \frac{1}{2}(-1)^{n+1}x^{2n},$$

$$f(x) = \frac{\pi}{2} + \sum_{n=0}^{\infty} \frac{1}{2}(-1)^{n+1}$$

6. ④

$$dy = dz + dz\ln\frac{z}{x} + z \cdot \frac{x}{z}\frac{xdz - zdx}{x^2},$$

hence grad $z = (1/2, 1/2)$.

7. ⑥ $(0,0), (2,2)$ - critical points,

$$d^2f = -6dx^2 + 12dxdy + (6-12y)dy^2,$$

$(2,2)$ is the maximum point, $f(2,2) = 8$, no extrema at $(0,0)$.