Academic year: 2020–2021, Semester: 2

Variant 1

1. ⑦ The 1st derivative is $y' = -\frac{(x+2)(x+18)}{(x-6)^4}$.
Local maximum is at $(-2, 0)$, local minimum is at $(-18, -1/54)$.
The 2nd derivative is $y'' = \frac{2(x^2+36x+132)}{(x-6)^5}$. Inflections are at $x = -18 \pm 8\sqrt{3}$.
Horizontal asymptote is $y = 0$, vertical asymptote is $x = 6$.

2. ⑤

$$\frac{2x^3 - 7x^2 + 12x - 27}{(x-2)(x^2 - x + 3)} = 2 - \frac{3}{x-2} + \frac{2x+3}{x^2 - x + 3}.$$

Answer: $2x - 3\ln|x-2| + \ln(x^2 - x + 3) + \frac{8}{\sqrt{11}} \arctan \frac{2x-1}{\sqrt{11}} + C$.

3. ⑤ $f(x) \sim \dfrac{1}{x^{1/2-\alpha/3}}$, $x \to +0$; $f(x) \sim Cx^{3\alpha}$, $x \to +\infty$; $-\dfrac{3}{2} < \alpha < -\dfrac{1}{3}$.

4. ⑤ Converges as

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{2}{\left(1 + \frac{1}{n^2}\right)^{n^2+1}} = \frac{2}{e} < 1.$$

5. ⑦

- Pointwise convergent to $f = 0$ as

$$\lim_{n\to\infty} f_n(x) = 0, \quad x \in (0, \infty).$$

- Converges uniformly for $x \in (1, \infty)$ since we have $|f_n(x)| \le \frac{C}{n+1}$, as $\lim_{n\to\infty} \frac{\ln^2 x}{x} = 0$.

- There is no uniform convergence for $x \in (0, 1)$ since $\lim_{n\to\infty} f_n(\frac{1}{n}) = \lim_{n\to\infty} \frac{\ln^2 \frac{1}{n}}{\frac{2}{n}} = \infty$.

6. ⑦

- Pointwise convergent because $\left|\frac{x \sin \frac{x}{n}}{x+n}\right| \le \frac{x^2}{n^2}$.

- Converges uniformly for $x \in (0, 1)$ by the Weierstrass test as

$$\left|\frac{x \sin \frac{x}{n}}{x+n}\right| \le \frac{1}{n^2}.$$

- There is no uniform convergence for $x \in (1, \infty)$ since for $x = n$ we have $u_n(n) = \frac{1}{2} \sin 1 \nrightarrow 0$.

Variant 2

1.⑦ The 1st derivative is $y' = \frac{(x-6)^2(x+18)}{(x+2)^3}$. Local maximum is at $(-18, -54)$.
The 2nd derivative is $y'' = \frac{378(x-6)}{(x+2)^4}$. Inflection is at $x = 6$.
Oblique asymptote is $y = x - 22$, vertical asymptote is $x = -2$.

2.⑤

$$\frac{3x^3 + 13x^2 + 27x + 18}{(x+3)(x^2+x+2)} = 3 - \frac{27}{8(x+3)} + \frac{35(4x+3)}{32(x^2+x+2)}.$$

Answer: $3x - \frac{27}{8}\ln|x+3| + \frac{35}{16}\ln(x^2+x+2) + \frac{1}{8\sqrt{7}}\arctan\frac{2x+1}{\sqrt{7}} + C.$

3.⑤ $f(x) \sim \dfrac{1}{x^{1-\alpha/2}}$, $x \to +0$; $f(x) \sim \dfrac{1}{\ln x \cdot x^{3/2-2\alpha}}$, $x \to +\infty$; $0 < \alpha < \dfrac{1}{4}$.

4.⑤ Converges as

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\cos\frac{1}{\sqrt{n}}\right)^n = e^{-1/2} < 1.$$

5.⑦

- Pointwise convergent to $f = 0$ as

$$\lim_{n\to\infty} f_n(x) = 0, \quad x \in (0, \infty).$$

- Converges uniformly for $x \in (1, \infty)$ since $|f_n(x)| \le \frac{\pi}{n+2}$.
- There is no uniform convergence for $x \in (0, 1)$ since $\lim_{n\to\infty} f_n(\frac{1}{n}) = \sin\frac{\pi}{3} \neq 0$.

6.⑦

- Pointwise convergent because $\sqrt{\frac{n}{x+1}}\left(e^{x^2/n^2} - 1\right) \sim \frac{x^2}{n^{3/2}\sqrt{x+1}}$.
- Converges uniformly for $x \in (0, 1)$ by the Weierstrass test as

$$|u_n(x)| \le \sqrt{n}\frac{2}{n^2}.$$

- There is no uniform convergence for $x \in (1, \infty)$ since for $x = n$ we have $\lim_{n\to\infty} u_n(n) = e - 1 \neq 0$.