A priori estimates of minimizers of an integral functional extending the variational Strong Maximum Principle

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Outline

- **Inf (Sup)-convolution** with a gauge function
- Variational properties of the Inf (Sup)-convolution
- **Strong Maximum Principle.** Traditional settings
- Variational generalizations. From harmonic functions to convolutions
- The problem with a linear perturbation
- **A priori estimates** of minimizers near nonextremum points
- "Cross-gap" extension of the Variational Strong Maximum Principle
- Back to the limit case. Example
- References
Inf (Sup)-convolution with a gauge function

This is the continuation of my talk given 2 years ago at the IFIP Conference in Klagenfurt dedicated to various properties of Inf-Convolutions

So, let me start with basic notations. Given a nonempty closed set $C \subset \mathbb{R}^n$ and $a > 0$ the Inf-convolution of a function $\theta : C \to \mathbb{R}$ with a gauge function $\rho_{F^0}(\cdot)$ is defined as

$$u_{\theta,F}^+(x) := \inf_{y \in C} \{ \theta(y) + a \rho_{F^0}(x - y) \} \quad (1)$$

Here $F$ is a compact convex subset of $\mathbb{R}^n$ with $0 \in \text{int} F$ (a gauge), $F^0$ means the polar set for $F$ and

$$\rho_{F^0}(\xi) := \inf \{ \lambda > 0 : \xi \in \lambda F^0 \}$$

Similarly, the Sup-convolution is

$$u_{\theta,F}^-(x) := \sup_{y \in C} \{ \theta(y) - a \rho_{F^0}(y - x) \} \quad (2)$$
Inf (Sup)-convolution with a gauge function

Under some supplementary condition on \( \theta \) or/and \( C \) the Inf (Sup)-convolution above can be seen as a viscosity solution of a Hamilton-Jacobi-Bellman equation, or as the Minimal Time necessary to achieve \( C \) by trajectories of some associated differential inclusion.

We are interested instead in the "minimal" (respectively, "maximal") property of this function among all the minimizers of some "elliptic" Variational Problem.
Variational properties of the Inf (Sup)-convolution

Namely, given an open bounded connected region $\Omega \subset \mathbb{R}^n$ containing $C$ and a lower semicontinuous convex function $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ with $f(0) = 0$ let us consider the integral functional

$$u(\cdot) \mapsto \int_{\Omega} f(\rho_F(\nabla u(x))) \, dx \tag{3}$$

In order to formulate the first (comparison or extension) result we assume also $\Omega$ to be convex and the boundary $\partial F$ to be smooth.

Denote by

$$a := \sup f^{-1}(0)$$

Consider a closed (not necessarily convex) subset $C \subset \Omega$ and a function $\theta(\cdot)$ defined on $C$ with a kind of slope condition:

$$\theta(x) - \theta(y) \leq a\rho_{F_0}(x - y), \quad x, y \in C \tag{4}$$
Then the following result holds

**Theorem A**

The Inf-convolution $u_{\theta,F}^+(\cdot)$ is a minimizer of the functional (3) on $u_0(\cdot) + W_{0,1}^{1,1}(\Omega)$, which is maximal among all the minimizers with the same boundary condition. In other words,

If $\bar{u}(\cdot)$ is a continuous admissible minimizer of (3) on $u_0(\cdot) + W_{0,1}^{1,1}(\Omega)$ such that

\begin{enumerate}
  \item \( \bar{u}(x) = u_{\theta,F}^+(x) = \theta(x) \quad \forall x \in C \)
  \item \( \bar{u}(x) \geq u_{\theta,F}^+(x) \quad \forall x \in \Omega \)
\end{enumerate}

then

$$\bar{u}(x) \equiv u_{\theta,F}^+(x), \quad x \in \Omega$$
Similarly,

**Theorem A**

The Sup-convolution $u_{\theta,F}^\leftarrow(\cdot)$ is a minimizer of the functional (3) on $u_0(\cdot) + W_0^{1,1}(\Omega)$ for a given boundary function $u_0(\cdot)$, which is minimal among all the minimizers in the following sense:

If $\bar{u}(\cdot)$ is a continuous admissible minimizer of (3) on $u_0(\cdot) + W_0^{1,1}(\Omega)$ such that

1. $\bar{u}(x) = u_{\theta,F}^\leftarrow(x) = \theta(x) \quad \forall x \in C$
2. $\bar{u}(x) \leq u_{\theta,F}^\leftarrow(x) \quad \forall x \in \Omega$

then

$$\bar{u}(x) \equiv u_{\theta,F}^\leftarrow(x), \quad x \in \Omega$$
The results above can be considered as a natural extension of the **Strong Maximum Principle** for elliptic equations/variational problems.

Let us follow the way from the classic setting for harmonic functions to Theorems above.

The **Strong Maximum Principle** (SMP) for the harmonic functions (solutions of the Laplace equation $\Delta u = 0$):

if a harmonic (in some open connected domain $\Omega \subset \mathbb{R}^n$) function attends its minimum (maximum) in an interior point $x_0 \in \Omega$ then it is constant.
Strong Maximum Principle. Traditional settings

Next, since $\Delta u = 0$ is the Euler-Lagrange equation in the problem of minimization of the functional

$$
\frac{1}{2} \int_{\Omega} \|\nabla u(x)\|^2 \, dx,
$$

one can formulate the Variational Strong Maximum Principle by substituting in the place of $\frac{1}{2} t^2$ in the functional above any convex lower semicontinuous function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ with $f(0) = 0$

So we have the rotationally invariant functional

$$
\int_{\Omega} f \left( \|\nabla u(x)\| \right) \, dx \tag{5}
$$
Then, the Variational SMP can be formulated as follows.

Given an open bounded connected region $\Omega \subset \mathbb{R}^n$ there is no continuous nonconstant minimizer of the functional (5) on $u_0(\cdot) + W_0^{1,1}(\Omega)$ admitting its minimal (maximal) value in $\Omega$,

or, equivalently,

if a continuous nonnegative (nonpositive) minimizer $u(\cdot)$ of (5) on $u_0(\cdot) + W_0^{1,1}(\Omega)$ touches zero at some point $x_0 \in \Omega$ then $u(x) \equiv 0$. 


Strong Maximum Principle. Traditional settings

The necessary and sufficient conditions guaranteeing the validity of the Variational SMP for (5) were proposed by A. Cellina in 2002:

\[(H_1) \quad \partial f^*(0) = \{0\} \text{ (strict convexity at 0)}\]
\[(H_2) \quad \partial f(0) = \{0\} \text{ (smoothness at 0)}\]

Here \(f^*\) means the Legendre-Fenchel transform of \(f\)
First we slightly extend the Cellina’s result to the functional with a general gauge-type symmetry:

$$\int_{\Omega} f(\rho_F(\nabla u(x))) \, dx$$  \hspace{2cm} (6)

where $F \subset \mathbb{R}^n$ is a closed convex bounded set with $0 \in \text{int} F$, and $\rho_F(\cdot)$ is the Minkowski functional associated to $F$. 

Then we assume that the hypothesis \((H_1)\) fails, i.e.,

\[
\partial f^*(0) = f^{-1}(0) = [-a, a]
\]

for some \(a > 0\) (the function \(f(\cdot)\) is extended symmetrically to \(\mathbb{R}\)).

In this case SMP in the traditional sense is obviously wrong, but it can be extended by considering in the place of the identical zero (or constant) another "test" function, namely,

\[
\theta + a \rho_{F_0}(x - x_0)
\]

where \(\theta \in \mathbb{R}\) and \(x_0 \in \Omega\).

This is a particular case of the Inf-convolution when the set \(C = \{x_0\}\).
We have the following local result

Let $\Omega \subset \mathbb{R}^n$ and a continuous minimizer $\bar{u}(\cdot)$ of the functional (6) on $u_0(\cdot) + W^{1,1}_0(\Omega)$ be such that for some $x_0 \in \Omega$ and $\delta > 0$

$$\bar{u}(x) \geq \bar{u}(x_0) + a\rho_{F^0}(x - x_0) \quad \forall x \in x_0 + \delta F^0 \subset \Omega$$

Then

$$\bar{u}(x) = \bar{u}(x_0) + a\rho_{F^0}(x - x_0) \quad \forall x \in x_0 + \frac{\delta}{\|F\| \|F^0\| + 1} F^0$$

Here $\|F\|$ and $\|F^0\|$ are the radii of the spheres centred at the origin circumscribed around $F$ and inscribed into $F$, respectively
Variational generalizations

The previous result can be formulated *globally* whenever the gauge $F$ has smooth boundary (equivalently, $\rho_F(\xi)$ is differentiable at each $\xi \neq 0$, or $F^0$ is rotund) and the region $\Omega$ is *densely star-shaped* w.r.t. $x_0 \in \Omega$ (in particular, $\Omega$ can be *convex*).

$\Omega \in \mathbb{R}^n$ is said to be *densely star-shaped* w.r.t. $x_0 \in \Omega$ if $\Omega \subset \overline{\text{Star}_\Omega(x_0)}$ where

$$\text{Star}_\Omega(x_0) := \{ x : \lambda x + (1 - \lambda)x_0 \in \Omega \text{ for all } \lambda \in [0, 1] \}$$
Namely, under these hypotheses

the inequality

\[ \bar{u}(x) \geq \bar{u}(x_0) + a\rho F_0(x - x_0) \quad \forall x \in \Omega \]

implies that

\[ \bar{u}(x) = \bar{u}(x_0) + a\rho F_0(x - x_0) \quad \forall x \in \Omega \]

Observe that if \( a = 0 \) then one immediately obtains from above the traditional Variational SMP (although under some suplementary assumptions such as the star-sharpness of \( \Omega \))

Simple examples show that the latter assumption can not be avoided
Considering in the place of a singleton \( \{x_0\} \) an arbitrary nonempty closed subset \( C \subset \Omega \) and in the place of a real number \( \theta \) a Lipschitz continuous function \( \theta(\cdot) \) defined on \( C \) and satisfying the slope condition (4) the generalized Strong Maximum Principle admits form of an Extremal Extention Principle (from a barrier inside \( \Omega \)) as formulated above.

But for this we need to assume the set \( \Omega \) to be convex and \( F \) to have smooth boundary.
The problem with a linear perturbation

Let us extend now the SMP in another way by considering the same "elliptic" integral functional but with linear perturbations:

\[ \int_{\Omega} \left[ f \left( \rho \mathcal{F}(\nabla u(x)) \right) + \sigma u(x) \right] \, dx \]  \hspace{1cm} (7)

or

\[ \int_{\Omega} \left[ f \left( \rho \mathcal{F}(\nabla u(x)) \right) - \sigma u(x) \right] \, dx \]  \hspace{1cm} (8)

with some real \( \sigma > 0 \)

In the case \( f \circ \rho \mathcal{F}(\xi) = \frac{1}{2} \|\xi\| \) the Euler-Lagrange equation for the problems (7) and (8) is the Poisson equation (with the constant non zero right-hand side)

\[ \Delta u(x) = \pm \sigma \]
The problem with a linear perturbation

We are interested in the following: how far the minimizers in (7) and (8) defer from the minimizers in the homogeneous case (with $\sigma = 0$) in their extreme properties.

In other words, we want to obtain some estimates of minimizers in a neighbourhood of their local minimum or maximum points that in the limit case ($\sigma = 0$) would turn back the standard SMP (or its extensions).
The problem with a linear perturbation

Indeed, such estimates can be made by using a class of a priori minimizers in the problems (7) and (8) found by A. Cellina in 2007.

Namely A. Cellina proved that the functions

$$\omega_{x_0,k}^+(x) := \frac{n}{\sigma} f^* \left( \frac{\sigma}{n} \rho F_0 (x - x_0) \right) + k$$

(respectively,

$$\omega_{x_0,k}^-(x) := -\frac{n}{\sigma} f^* \left( \frac{\sigma}{n} \rho F_0 (x_0 - x) \right) + k$$)

$x_0 \in \Omega$, $k \in \mathbb{R}$, minimize the functional (7) (respectively, (8)) on the class of Sobolev functions with the same boundary conditions.

Here $f^*(\cdot)$ means the Legendre-Fenchel conjugate of $f(\cdot)$ (symmetrically extended to $\mathbb{R}$).
The problem with a linear perturbation

In order to be defined on whole $\Omega$ the following inclusions, naturally, should hold:

$$\Omega \subset x_0 \pm b \cdot \frac{n}{\sigma} F^0$$  \hspace{1cm} (9)

where $b := \sup(\text{dom} f^*)$ (equivalently, $b > 0$ is such that $f(\cdot)$ is affine on $[b, +\infty]$)

So, if $\Omega$ is enough small (or $b > 0$ is enough large, in particular, if $b = +\infty$, i.e., $f(\cdot)$ never becomes affine) then $x_0$ satisfying the conditions (9) exist
A priori estimates of minimizers near nonextremum points

First, by using the solutions $\omega_{x_0, \kappa}(\cdot)$ we obtain the following estimates of minimizers near their nonextremum points.

Let $u(\cdot)$ be a continuous minimizer in the problem (7), $\beta > 0$ be small enough and $x_0 \in \Omega$ satisfying the inclusion (9) be not a local maximum point of $u(\cdot)$ in the sense that

$$u(x_0) < k - \frac{n}{\sigma} f^* \left( \frac{\sigma}{n} \beta \right)$$

where

$$k := \max_{x \in x_0 + \beta F^0} u(x)$$

Then

$$u(x) \leq k - \frac{n}{\sigma} \left[ f^* \left( \frac{\sigma}{n} \beta \right) - f^* \left( \frac{\sigma}{n} \rho F^0(x - x_0) \right) \right]$$

for all $x \in x_0 + \beta F^0 \subset \Omega$ (i.e., $\rho F^0(x - x_0) \leq \beta$)
Similarly,

If $u(\cdot)$ is a continuous minimizer in (8), $\beta > 0$ is small enough and $x_0 \in \Omega$ is not a local minimum of $u(\cdot)$ and, moreover,

$$u(x_0) > k + \frac{n}{\sigma} f^* \left( \frac{\sigma}{n \beta} \right)$$

where

$$k := \min_{x \in x_0 - \beta F^0} u(x),$$

then

$$u(x) \geq k + \frac{n}{\sigma} \left[ f^* \left( \frac{\sigma}{n \beta} \right) - f^* \left( \frac{\sigma}{n} \rho F^0 (x_0 - x) \right) \right]$$

for all $x \in x_0 - \beta F^0 \subset \Omega \ (\rho F^0 (x_0 - x) \leq \beta)$
"Cross-gap" extension of the Variational SMP

Finally, from the above estimates we deduce the following estimate

Let \( u(\cdot) \) be a continuous minimizer in the problem (7). If \( x_0 \in \Omega \) is a local maximum point of \( u(\cdot) \), namely, there is \( \delta > 0 \) with \( u(x_0) \geq u(x) \) for all \( x \in x_0 - \delta F^0 \subset \Omega \). Then, in a slightly smaller neighbourhood of \( x_0 \) (namely, in \( x_0 - \frac{\delta}{1+\|F\|\|F^0\|} F^0 \)) the inequality

\[
    u(x) \geq u(x_0) - \frac{n}{\sigma} f^* \left( \frac{\sigma}{n} \rho F^0(x_0 - x) \right)
\]

holds
Cross-gap” extension of the Variational SMP

And, symmetrically,

- If \( u(\cdot) \) is a \textbf{continuous minimizer} in the problem (8) and \( x_0 \in \Omega \) is a \textbf{local minimum} point of \( u(\cdot) \) (with some \( \delta > 0 \) we have \( u(x_0) \leq u(x) \) for all \( x \in x_0 + \delta F^0 \subset \Omega \)), then

\[
u(x) \leq u(x_0) + \frac{n}{\sigma} f^* \left( \frac{\sigma}{n} \rho F^0 (x - x_0) \right)
\]

for all \( x \in x_0 + \frac{\delta}{1 + \|F\|\|F^0\|} F^0 \)

Observe that the right-hand side of the inequality (10) (for a minimizer of the functional with the \textbf{positive perturbation} \( \sigma u(x) \)) is exactly \( \omega_{x_0.u(x_0)}(x) \) (a Cellina’s minimizer of the functional with the \textbf{negative perturbation} \( -\sigma u(x) \)) and \textbf{vice versa}
"Cross-gap" extension of the Variational SMP

So, close to extremum points of a continuous minimizer \( u(\cdot) \) (maximum or minimum, respectively, depending on sign of the perturbation) we have a gap between \( u(x_0) \) and

\[
u(x_0) + \frac{n}{\sigma} f^* \left( \frac{\sigma}{n} \rho F_0(x - x_0) \right),
\]
or

\[
u(x_0) - \frac{n}{\sigma} f^* \left( \frac{\sigma}{n} \rho F_0(x_0 - x) \right)
\]

containing the values of \( u(\cdot) \)
Back to the limit case. Example

This "cross-gap" shrinks to zero (i.e., to validity of the traditional SMP) as \( \sigma \to 0 \pm \) whenever the conjugate \( f^* \) is differentiable at the origin and \( (f^*)'(0) = 0 \) (equivalently, \( f \) is strictly convex at the origin, or \( f^{-1}(0) = \{0\} \))

If, instead, \( f^{-1}(0) = [-a, a], \ a > 0 \), then \( f^* \) is affine (\( = \pm at \)) near the origin. Therefore, the "cross-gap" is reduced to the interval between \( u(x_0) \) and

\[
u(x_0) \pm a \rho F^0 (\pm (x - x_0))
\]

This, in turn, easily implies the (local) extreme properties of the convolutions (with \( C = \{x_0\} \)) what we talked about at the beginning.
Let $p > 1$ and $q > 1$ be its conjugate number, i.e., $1/p + 1/q = 1$. Consider the function $f(t) = \frac{t^p}{p}$, $t \geq 0$, which satisfies all the standing assumptions and whose conjugate is $f^*(t) = \frac{t^q}{q}$, $t \geq 0$

Let us given a **continuous minimizer** $u(\cdot)$ in the problem

$$
\text{Minimize } \left\{ \int_{\Omega} \left[ \frac{1}{p} \| \nabla u(x) \|^p + \sigma u(x) \right] \, dx : u(\cdot) \in \bar{u}(\cdot) + W^{1,1}_0(\Omega) \right\}
$$

Assuming that $x_0 \in \Omega$ is a **local maximum** point of $u(\cdot)$ from the above result we conclude that near $x_0$ this minimizer satisfies a **lower estimate**

$$
u(x_0) \geq u(x) \geq u(x_0) - \frac{1}{q} n \left( \frac{\sigma}{n} \| x - x_0 \| \right)^q
$$

$$
= u(x_0) - \left( \frac{\sigma}{n} \right)^{q-1} \| x - x_0 \|^q
$$
Back to the limit case. Example

Symmetrically,

If \( u(\cdot) \) is a continuous minimizer in

\[
\text{Minimize} \left\{ \int_{\Omega} \left[ \frac{1}{p} \| \nabla u(x) \|^p - \sigma u(x) \right] \, dx : u(\cdot) \in \bar{u}(\cdot) + \mathcal{W}^{1,1}_0(\Omega) \right\},
\]

and \( x_0 \in \Omega \) is a local minimum point of \( u(\cdot) \) we have the following (upper) estimate, which holds in a neighbourhood of \( x_0 \):

\[
u(x_0) \leq u(x) \leq u(x_0) + \left( \frac{\sigma}{n} \right)^{q-1} \| x - x_0 \|^q
\]

A. Cellina, Uniqueness and comparison results for functionals depending on $u$ and $\nabla u$, *SIAM J. on Control and Optim.* **46** (2007), 711-716

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