A priori estimates of minimizers of an integral functional extending the variational Strong Maximum Principle

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Inf (Sup)-convolution with a gauge function

This is the continuation of my talk given 2 years ago at the IFIP Conference in Klagenfurt dedicated to varios properties of Inf-Convolutions

So, let me start with basic notations. Given a nonempty closed set $C \subset \mathbb{R}^n$ and $a > 0$ the Inf-convolution of a function $\theta : C \to \mathbb{R}$ with a gauge function $\rho_{F^0}(\cdot)$ is defined as

$$u^{+}_{\theta,F}(x) := \inf_{y \in C} \{\theta(y) + a\,\rho_{F^0}(x-y)\} \qquad (1)$$

Here F is a compact convex subset of $\mathbb{R}^n$ with $0 \in \operatorname{int} F$ (a gauge), F^0 means the polar set for F and

$$\rho_{F^0}(\xi) := \inf\{\lambda > 0 : \xi \in \lambda F^0\}$$

Similarly, the Sup-convolution is

$$u^{-}_{\theta,F}(x) := \sup_{y \in C} \{\theta(y) - a\,\rho_{F^0}(y-x)\} \qquad (2)$$

Inf (Sup)-convolution with a gauge function

Under some supplementary condition on θ or/and C the Inf (Sup)-convolution above can be seen as a viscosity solution of a Hamilton-Jacobi-Bellman equation, or as the Minimal Time necessary to achieve C by trajectories of some associated differential inclusion

We are interested instead in the "minimal" (respectively, "maximal") property of this function among all the minimizers of some "elliptic" Variational Problem

Variational properties of the Inf (Sup)-convolution

Namely, given an open bounded connected region $\Omega \subset \mathbb{R}^n$ containing C and a lower semicontinuous convex function $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ with $f(0) = 0$ let us consider the integral functional

$$u(\cdot) \mapsto \int_\Omega f(\rho_F(\nabla u(x)))\, dx \tag{3}$$

In order to formulate the first (comparison or extension) result we assume also Ω to be convex and the boundary ∂F to be smooth

Denote by

$$a := \sup f^{-1}(0)$$

Consider a closed (not necessarily convex) subset $C \subset \Omega$ and a function $\theta(\cdot)$ defined on C with a kind of slope condition:

$$\theta(x) - \theta(y) \leq a\rho_{F^0}(x-y), \quad x, y \in C \tag{4}$$

Variational properties of the Inf (Sup)-convolution

Then the following result holds

Theorem A

The Inf-convolution $u^+_{\theta,F}(\cdot)$ is a minimizer of the functional (3) on $u_0(\cdot) + \mathbf{W}_0^{1,1}(\Omega)$, which is maximal among all the minimizers with the same boundary condition. In other words,

If $\bar{u}(\cdot)$ is a continuous admissible minimizer of (3) on $u_0(\cdot) + \mathbf{W}_0^{1,1}(\Omega)$ such that

(i) $\bar{u}(x) = u^+_{\theta,F}(x) = \theta(x) \qquad \forall x \in C$

(ii) $\bar{u}(x) \geq u^+_{\theta,F}(x) \qquad \forall x \in \Omega$

then

$$\bar{u}(x) \equiv u^+_{\theta,F}(x), \qquad x \in \Omega$$

Variational properties of the Inf (Sup)-convolution

Similarly,

Theorem A'

The Sup-convolution $u^{-}_{\theta,F}(\cdot)$ is a minimizer of the functional (3) on $u_0(\cdot) + \mathbf{W}_0^{1,1}(\Omega)$ for a given boundary function $u_0(\cdot)$, which is minimal among all the minimizers in the following sense:

If $\bar{u}(\cdot)$ is a continuous admissible minimizer of (3) on $u_0(\cdot) + \mathbf{W}_0^{1,1}(\Omega)$ such that

(i) $\bar{u}(x) = u^{-}_{\theta,F}(x) = \theta(x) \qquad \forall x \in C$

(ii) $\bar{u}(x) \leq u^{-}_{\theta,F}(x) \qquad \forall x \in \Omega$

then

$$\bar{u}(x) \equiv u^{-}_{\theta,F}(x), \qquad x \in \Omega$$

Strong Maximum Principle. Traditional settings

The results above can be considered as a natural extension of the Strong Maximum Principle for elliptic equations/variational problems

Let us follow the way from the classic setting for harmonic functions to Theorems above

The Strong Maximum Principle (SMP) for the harmonic functions (solutions of the Laplace equation $\Delta u = 0$):

if a harmonic (in some open connected domain $\Omega \subset \mathbb{R}^n$) function attends its minimum (maximum) in an interior point $x_0 \in \Omega$ then it is constant

Strong Maximum Principle. Traditional settings

Next, since $\Delta u = 0$ is the Euler-Lagrange equation in the problem of minimization of the functional

$$1/2\int_{\Omega} \|\nabla u(x)\|^2 \, dx,$$

one can formulate the Variational Strong Maximum Principle by substituting in the place of $\frac{1}{2}t^2$ in the functional above any convex lower semicontinuous function $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ with $f(0) = 0$

So we have the rotationally invariant functional

$$\int_{\Omega} f\left(\|\nabla u(x)\|\right) \, dx \tag{5}$$

Strong Maximum Principle. Traditional settings

Then, the Variational SMP can be formulated as follows

Given an open bounded connected region $\Omega \subset \mathbb{R}^n$ there is no continuous nonconstant minimizer of the functional (5) on $u_0(\cdot) + \mathbf{W}_0^{1,1}(\Omega)$ admiting its minimal (maximal) value in Ω,

or, equivalently,

if a continuous nonnegative (nonpositive) minimizer $u(\cdot)$ of (5) on $u_0(\cdot) + \mathbf{W}_0^{1,1}(\Omega)$ touches zero at some point $x_0 \in \Omega$ then $u(x) \equiv 0$

Strong Maximum Principle. Traditional settings

The necessary and sufficient conditions guaranteeing the validity of the Variational SMP for (5) were proposed by A. Cellina in 2002:

(H_1) $\partial f^*(0) = \{0\}$ (strict convexity at 0)

(H_2) $\partial f(0) = \{0\}$ (smoothness at 0)

Here f^* means the Legendre-Fenchel transform of f

Variational generalizations

First we slightly extend the Cellina's result to the functional with a general gauge-type symmetry:

$$\int_{\Omega} f\left(\rho_F\left(\nabla u\left(x\right)\right)\right)\,dx \tag{6}$$

where $F \subset \mathbb{R}^n$ is a closed convex bounded set with $0 \in \mathrm{int}F$, and $\rho_F(\cdot)$ is the Minkowski functional associated to F

Variational generalizations

Then we assume that the hypothesis (H_1) fails, i.e.,

$$\partial f^*(0) = f^{-1}(0) = [-a, a]$$

for some $a > 0$ (the function $f(\cdot)$ is extended symmetrically to $\mathbb{R}$)

In this case SMP in the traditional sense is obviously wrong, but it can be extended by considering in the place of the identical zero (or constant) another "test" function, namely,

$$\theta + a\,\rho_{F^0}(x - x_0)$$

where $\theta \in \mathbb{R}$ and $x_0 \in \Omega$

This is a particular case of the Inf-convolution when the set $C = \{x_0\}$

Variational generalizations

We have the following local result

Let $\Omega \subset \mathbb{R}^n$ and a continuous minimizer $\bar{u}(\cdot)$ of the functional (6) on $u_0(\cdot) + \mathbf{W}_0^{1,1}(\Omega)$ be such that for some $x_0 \in \Omega$ and $\delta > 0$

$$\bar{u}(x) \geq \bar{u}(x_0) + a\rho_{F^0}(x - x_0) \qquad \forall x \in x_0 + \delta F^0 \subset \Omega$$

Then

$$\bar{u}(x) = \bar{u}(x_0) + a\rho_{F^0}(x - x_0) \qquad \forall x \in x_0 + \frac{\delta}{\|F\| \, \|F^0\| + 1} F^0$$

Here $\|F\|$ and $\|F^0\|$ are the radii of the spheres centred at the origin circumscribed around F and inscribed into F, respectively

Variational generalizations

The previous result can be formulated globally whenever the gauge F has smooth boundary (equivalently, $\rho_F(\xi)$ is differentiable at each $\xi \neq 0$, or F^0 is rotund) and the region Ω is densely star-shaped w.r.t. $x_0 \in \Omega$ (in particular, Ω can be convex)

$\Omega \in \mathbb{R}^n$ is said to be densely star-shaped w.r.t. $x_0 \in \Omega$ if $\Omega \subset \overline{\mathrm{Star}_\Omega(x_0)}$ where

$$\mathrm{Star}_\Omega(x_0) := \{x : \lambda x + (1-\lambda)x_0 \in \Omega \ \text{ for all } \ \lambda \in [0,1]\}$$

Variational generalizations

Namely, under these hypotheses

the inequality

$$\bar{u}(x) \geq \bar{u}(x_0) + a\rho_{F^0}(x - x_0) \qquad \forall x \in \Omega$$

implies that

$$\bar{u}(x) = \bar{u}(x_0) + a\rho_{F^0}(x - x_0) \qquad \forall x \in \Omega$$

Observe that if $a = 0$ then one immediately obtains from above the traditional Variational SMP (although under some suplementary assumptions such as the star-sharpness of Ω)

Simple examples show that the latter assumption can not be avoided

Variational generalizations

Considering in the place of a singleton $\{x_0\}$ an arbitrary nonempty closed subset $C \subset \Omega$ and in the place of a real number θ a Lipschitz continuous function $\theta(\cdot)$ defined on C and satisfying the slope condition (4) the generalized Strong Maximum Principle admits form of an Extremal Extention Principle (from a barrier inside Ω) as formulated above

But for this we need to assume the set Ω to be convex and F to have smooth boundary

The problem with a linear perturbation

Let us extend now the SMP in another way by considering the same "elliptic" integral functional but with linear perturbations:

$$\int_{\Omega} \left[f\left(\rho_F(\nabla u(x))\right) + \sigma u(x) \right] dx \tag{7}$$

or

$$\int_{\Omega} \left[f\left(\rho_F(\nabla u(x))\right) - \sigma u(x) \right] dx \tag{8}$$

with some real $\sigma > 0$

In the case $f \circ \rho_F(\xi) = \frac{1}{2}\|\xi\|$ the Euler-Lagrange equation for the problems (7) and (8) is the Poison equation (with the constant non zero right-hand side)

$$\Delta u(x) = \pm\sigma$$

The problem with a linear perturbation

We are interested in the following: how far the minimizers in (7) and (8) defer from the minimizers in the homogeneous case (with $\sigma = 0$) in their extreme properties

In other words, we want to obtain some **estimates** of minimizers in a neighbourhood of their local minimum or maximum points that in the limit case ($\sigma = 0$) would turn back the standart SMP (or its extensions)

The problem with a linear perturbation

Indeed, such estimates can be made by using a class of *a priori* minimizers in the problems (7) and (8) found by A. Cellina in 2007

Namely A. Cellina proved that the functions

$$\omega^{+}_{x_0,k}(x) := \frac{n}{\sigma} f^{*}\left(\frac{\sigma}{n}\rho_{F^0}(x - x_0)\right) + k$$

(respectively,

$$\omega^{-}_{x_0,k}(x) := -\frac{n}{\sigma} f^{*}\left(\frac{\sigma}{n}\rho_{F^0}(x_0 - x)\right) + k),$$

$x_0 \in \Omega, k \in \mathbb{R}$, minimize the functional (7) (respectively, (8)) on the class of Sobolev functions with the same boundary conditions

Here $f^{*}(\cdot)$ means the Legendre-Fenchel conjugate of $f(\cdot)$ (symmetrically extended to $\mathbb{R}$)

The problem with a linear perturbation

In order to be defined on whole Ω the following inclusions, naturally, should hold:

$$\Omega \subset x_0 \pm b \cdot \frac{n}{\sigma} F^0 \tag{9}$$

where $b := \sup(\mathrm{dom} f^*)$ (equivalently, $b > 0$ is such that $f(\cdot)$ is affine on $[b, +\infty[$)

So, if Ω is enough small (or $b > 0$ is enough large, in particular, if $b = +\infty$, i.e., $f(\cdot)$ never becomes affine) then x_0 satisfying the conditions (9) exist

A priori estimates of minimizers near nonextremum points

First, by using the solutions $\omega^{\pm}_{x_0,k}(\cdot)$ we obtain the following estimates of minimizers near their nonextremum points

Let $u(\cdot)$ be a continuous minimizer in the problem (7), $\beta > 0$ be small enough and $x_0 \in \Omega$ satisfying the inclusion (9) be not a local maximum point of $u(\cdot)$ in the sense that

$$u(x_0) < k - \frac{n}{\sigma} f^* \left(\frac{\sigma}{n} \beta \right)$$

where

$$k := \max_{x \in x_0 + \beta F^0} u(x)$$

Then

$$u(x) \leq k - \frac{n}{\sigma} \left[f^* \left(\frac{\sigma}{n} \beta \right) - f^* \left(\frac{\sigma}{n} \rho_{F^0}(x - x_0) \right) \right]$$

for all $x \in x_0 + \beta F^0 \subset \Omega$ (i.e., $\rho_{F^0}(x - x_0) \leq \beta$)

Similarly,

If $u(\cdot)$ is a continuous minimizer in (8), $\beta > 0$ is small enough and $x_0 \in \Omega$ is not a local minimum of $u(\cdot)$ and, moreover,

$$u(x_0) > k + \frac{n}{\sigma} f^* \left(\frac{\sigma}{n} \beta \right)$$

where

$$k := \min_{x \in x_0 - \beta F^0} u(x),$$

then

$$u(x) \geq k + \frac{n}{\sigma} \left[f^* \left(\frac{\sigma}{n} \beta \right) - f^* \left(\frac{\sigma}{n} \rho_{F^0}(x_0 - x) \right) \right]$$

for all $x \in x_0 - \beta F^0 \subset \Omega$ $(\rho_{F^0}(x_0 - x) \leq \beta)$

"Cross-gap" extension of the Variational SMP

Finally, from the above estimates we deduce the following estimate

- Let $u(\cdot)$ be a continuous minimizer in the problem (7). If $x_0 \in \Omega$ is a local maximum point of $u(\cdot)$, namely, there is $\delta > 0$ with $u(x_0) \geq u(x)$ for all $x \in x_0 - \delta F^0 \subset \Omega$. Then, in a slightly smaller neighbourhood of x_0 (namely, in $x_0 - \frac{\delta}{1+\|F\|\|F^0\|} F^0$) the inequality

$$u(x) \geq u(x_0) - \frac{n}{\sigma} f^* \left(\frac{\sigma}{n} \rho_{F^0}(x_0 - x) \right) \qquad (10)$$

holds

"Cross-gap" extension of the Variational SMP

And, symmetrically,

- If $u(\cdot)$ is a continuous minimizer in the problem (8) and $x_0 \in \Omega$ is a local minimum point of $u(\cdot)$ (with some $\delta > 0$ we have $u(x_0) \leq u(x)$ for all $x \in x_0 + \delta F^0 \subset \Omega$), then

$$u(x) \leq u(x_0) + \frac{n}{\sigma} f^* \left(\frac{\sigma}{n} \rho_{F^0}(x - x_0) \right) \tag{11}$$

for all $x \in x_0 + \frac{\delta}{1+\|F\|\|F^0\|} F^0$

Observe that the right-hand side of the inequality (10) (for a minimizer of the functional with the positive perturbation $\sigma u(x)$) is exactly $\omega^-_{x_0 . u(x_0)}(x)$ (a Cellina's minimizer of the functional with the negative perturbation $-\sigma u(x)$) and **vice versa**

"Cross-gap" extension of the Variational SMP

So, close to extremum points of a continuous minimizer $u(\cdot)$ (maximum or minimum, respectively, depending on sign of the perturbation) we have a gap between $u(x_0)$ and

$$u(x_0) + \frac{n}{\sigma} f^* \left(\frac{\sigma}{n} \rho_{F^0}(x - x_0) \right),$$

or

$$u(x_0) - \frac{n}{\sigma} f^* \left(\frac{\sigma}{n} \rho_{F^0}(x_0 - x) \right)$$

containing the values of $u(\cdot)$

Back to the limit case. Example

This "cross-gap" shrinks to zero (i.e., to validity of the traditional SMP) as $\sigma \to 0\pm$ whenever the conjugate f^* is differentiable at the origin and $(f^*)'(0) = 0$ (equivalently, f is stricly convex at the origin, or $f^{-1}(0) = \{0\}$)

If, instead, $f^{-1}(0) = [-a, a],\ a > 0$, then f^* is affine ($= \pm at$) near the origin. Therefore, the "cross-gap" is reduced to the interval between $u(x_0)$ and

$$u(x_0) \pm a\rho_{F^0}\left(\pm(x - x_0)\right)$$

This, in turn, easily implies the (local) extreme properties of the convolutions (with $C = \{x_0\}$) what we talked about at the beginning

Back to the limit case. Example

Let $p > 1$ and $q > 1$ be its conjugate number, i.e., $1/p + 1/q = 1$. Consider the function $f(t) = \frac{t^p}{p}$, $t \geq 0$, which satisfies all the standing assumptions and whose conjugate is $f^*(t) = \frac{t^q}{q}$, $t \geq 0$

Let us given a continuous minimizer $u(\cdot)$ in the problem

$$\text{Minimize} \left\{ \int_\Omega \left[\frac{1}{p} \|\nabla u(x)\|^p + \sigma u(x) \right] dx : u(\cdot) \in \bar{u}(\cdot) + \mathbf{W}_0^{1,1}(\Omega) \right\}$$

Assuming that $x_0 \in \Omega$ is a local maximum point of $u(\cdot)$ from the above result we conclude that near x_0 this minimizer satisfies a lower estimate

$$\begin{aligned} u(x_0) \geq u(x) &\geq u(x_0) - \frac{1}{q} \frac{n}{\sigma} \left(\frac{\sigma}{n} \|x - x_0\| \right)^q \\ &= u(x_0) - \left(\frac{\sigma}{n} \right)^{q-1} \|x - x_0\|^q \end{aligned}$$

Back to the limit case. Example

Symmetrically,

If $u(\cdot)$ is a continuous minimizer in

$$\text{Minimize} \left\{ \int\limits_{\Omega} \left[\frac{1}{p} \|\nabla u(x)\|^p - \sigma u(x) \right] dx : u(\cdot) \in \bar{u}(\cdot) + \mathbf{W}_0^{1,1}(\Omega) \right\},$$

and $x_0 \in \Omega$ is a local minimum point of $u(\cdot)$ we have the following (upper) estimate, which holds in a neighbourhood of x_0:

$$u(x_0) \le u(x) \le u(x_0) + \left(\frac{\sigma}{n} \right)^{q-1} \|x - x_0\|^q$$

References

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