SEPARATION THEOREMS FOR WEAKLY CONVEX SETS WITH RESPECT TO A NONSYMMETRIC SEMINORM

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Let $E$ be a real Banach space.

A set $M$ is called quasiball if $M$ is convex closed and $0 \in \text{int } M$.

The *Minkowski functional of the quasiball* $M$

$$\mu_M(x) = \inf \{ t > 0 \mid x \in tM \}$$

is the nonsymmetric seminorm.

The *M-distance* from a point $x \in E$ to the set $A \subset E$ is

$$\varrho_M(x, A) = \inf_{a \in A} \mu_M(x - a) =$$

$$= \inf \{ t > 0 \mid (x - tM) \cap A \neq \emptyset \}.$$

The *ball of radius* $R$ and center $a$ is

$$\mathcal{B}_R(a) = \{ x \in E : \|x - a\| \leq R \}.$$

The Minkowski functional of the unit ball $\mu_{\mathcal{B}_1(0)}(x) = \|x\|$.

The *distance* from the point $x \in E$ to the set $A \subset E$ is

$$\varrho(x, A) = \inf_{a \in A} \|x - a\| =$$

$$= \inf \{ t > 0 \mid (x - \mathcal{B}_t(0)) \cap A \neq \emptyset \}.$$
The $M$-projection of the point $x \in E$ on the set $A \subset E$ is called

$$P_M(x, A) = (x - \varrho_M(x, A)M) \cap A$$

The projection of the point $x$ on the set $A$ is called

$$P(x, A) = \varrho(x, A)B_1(x) \cap A.$$
The set of *unit normals* for a set $A \subset E$ at a point $a \in A$ is defined as

$$N^1(a, A) = \{ z \in E | \exists t > 0 : a \in P(a + tz, A), \|z\| = 1 \}.$$ 

The set of *unit $M$-normals* for a set $A \subset E$ at a point $a \in A$ is defined as

$$N^1_M(a, A) = \{ z \in E | \exists t > 0 : a \in P_M(a + tz, A), \mu_M(z) = 1 \}.$$
Clark, Stern, Wolenski (1995) and Bernard, Thibault, Zlateva (2006, 2011) considered the following notion in Hilbert and Banach spaces accordingly: a closed set $A \subset E$ is called $r$-prox-regular if

$$a \in P(a + rz, A) \quad \forall a \in A, \quad \forall z \in N^1(a, A).$$

A closed set $A \subset E$ is called weakly convex with respect to the quasiball $M \subset E$ if

$$a \in P_M(a + z, A) \quad \forall a \in A, \quad \forall z \in N^1_M(a, A).$$

$\mathcal{WC}(M)$ denotes the class of weakly convex sets with respect to the quasiball $M$. 
The *convexity modulus* of the quasiball $M \subset E$ is defined as

$$
\delta_M(\varepsilon) = \inf \left\{ 1 - \frac{\mu_M(x+y)}{2} \Big| x, y \in M, \|x - y\| \geq \varepsilon \right\}.
$$

A quasiball is called *uniformly convex* if $\delta_M(\varepsilon) > 0$ for any $\varepsilon > 0$.

If the ball $\mathcal{B}_1(0)$ in the space $E$ is uniformly convex, then the space $E$ is uniformly convex.

The following definition generalizes the notion of uniform convexity:

The set $M \subset E$ is called *boundedly uniformly convex*, if

$$
\delta_M(\varepsilon, d) > 0 \quad \forall d > 0 \quad \forall \varepsilon > 0,
$$

where

$$
\delta_M(\varepsilon, d) = \inf \left\{ 1 - \frac{\mu_M(x+y)}{2} \Big| x, y \in M \cap \mathcal{B}_d(0), \|x - y\| \geq \varepsilon \right\}.
$$
The smoothness modulus of the quasiball $M \subset E$ is defined as
\[
\beta_M(t) = \left\{ \frac{\mu_M(x + ty) + \mu_M(x - ty)}{2} - 1 \right| x \in \partial M, y \in \partial \mathcal{B}_1(0) \right\}, \quad t \geq 0.
\]

The quasiball is called uniformly smooth if $\lim_{t \to +0} \frac{\beta_M(t)}{t} = 0$.

If the ball $\mathcal{B}_1(0)$ in the space $E$ is uniformly smooth then the space $E$ is uniformly smooth.
Problem: \( P_M(x, A) = \emptyset \)

The set \( M \subset E \) is called parabolic if for any \( b \in E \) the set \( (b + \frac{1}{2} M) \setminus M \) is bounded.

Note that the epigraph of parabola is parabolic, the epigraph of hyperbola is not parabolic.
A closed convex set $M$ in a Banach space $E$ is called *generating* if for any set $X$ such that the set $A = \bigcap_{x \in X} (M + x)$ is not empty there exists a closed convex set $B$ such that $A + B = M$.

Some examples of generating sets:
1) All closed convex sets in $\mathbb{R}^2$
2) The ball in Hilbert space
We call a set \( C \) \( r \)-strongly convex if \( C = \bigcap_{x \in X} \mathcal{B}_r(x) \).

\( SC(M) \) denotes the class of closed convex sets such that \( \exists C_1 : C + C_1 = M \).

If \( C \in SC(\mathcal{B}_r(0)) \) then \( C \) is \( r \)-strongly convex.

The inverse implication holds if the \( \mathcal{B}_r(0) \) is a generating set in \( E \).
Theorem 1. (M.V. Balashov, G.E. Ivanov)

Let $E$ be a uniformly convex and uniformly smooth Banach space. The following conditions are equivalent:

1) for any closed set $A \in \mathcal{WC}(\mathcal{B}_R(0))$ and a $r$-strongly convex set $C$ (where $0 < r < R$) there exist points $b, c \in E$ such that

$$C \subset \mathcal{B}_r(c) \subset \text{int} \ \mathcal{B}_R(b) \subset E \setminus A.$$

2) the ball in $E$ is a generating set.
Let there be given a quasiball $M \subset E$. The set $A \subset E$ is called $M$-closed if for any $x \in E \setminus A$ the inequality $\rho_M(x, A) > 0$ holds.

The set $A \subset E$ is called $M$-quasibounded, if it is $M$-closed and

$$\sup_{a \in \partial A, \|a\| \leq d} \sup_{z \in N^1_M(a, A)} \|z\| < +\infty, \quad \forall d > 0.$$
**Theorem 2.**

Let $E$ be a Banach space and the quasiball $M \subset E$ be parabolic and boundedly uniformly convex. Let $0 < r < R$, the sets $A, C \subset E$ be closed, $A \in \text{WC}(RM)$, $C \in \text{SC}(-rM)$, $A + R\text{int}~M \neq E$. Let at least one of the following statements hold

1) $\varrho_M(C, A) > 0$ or
2) $\text{int}~C \neq \emptyset$, $A \cap \text{int}~C = \emptyset$ and the quasiball $M$ is uniformly smooth, the set $A$ is $M$-quasibounded.

Then there exist $a, c \in E$ such that $\text{int}~C \subset c - \text{int}~rM \subset a - \text{int}~RM \subset E \setminus A$.

**Remark 1.**

*In case of a finite dimensional space the smoothness condition of $E$ may be aborted.*
Applications

$A(t) \cap C(t)$

$A(t)$

$C(t)$
Applications

\[ A(t_1) \cap C(t_1) \]

\[ A(t_1) \]

\[ C(t_1) \]


Thank you for your attention!