SEPARATION THEOREMS FOR WEAKLY CONVEX SETS WITH RESPECT TO A NONSYMMETRIC SEMINORM

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Let E be a real Banach space.

A set M is called quasiball if M is convex closed and $0 \in \operatorname{int} M$.

The *Minkowski functional* of the quasiball M
$\mu_M(x) = \inf \left\{ t > 0 \middle| \, x \in tM \right\}$
is the nonsymmetric seminorm.

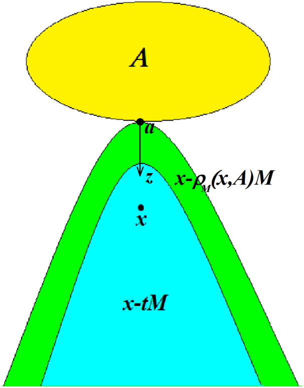
The M-*distance* from a point $x \in E$ to the set $A \subset E$ is
$\varrho_M(x, A) = \inf\limits_{a \in A} \mu_M(x - a) =$
$= \inf\{t > 0 \,|\, (x - tM) \bigcap A \neq \emptyset\}$.

The *ball* of radius R and center a is $\mathfrak{B}_R(a) = \{x \in E : \|x - a\| \leq R\}$.

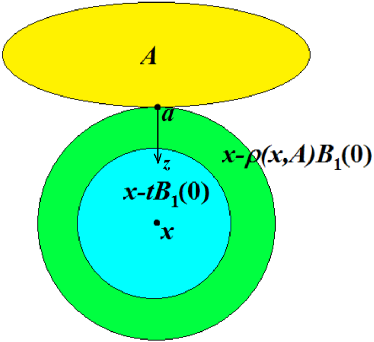
The Minkowski functional of the unit ball $\mu_{\mathfrak{B}_1(0)}(x) = \|x\|$.

The *distance* from the point $x \in E$ to the set $A \subset E$ is
$\varrho(x, A) = \inf\limits_{a \in A} \|x - a\| =$
$= \inf\{t > 0 \,|\, (x - \mathfrak{B}_t(0)) \bigcap A \neq \emptyset\}$.

The *M-projection* of the point $x \in E$ on the set $A \subset E$ is called
$P_M(x, A) = (x - \varrho_M(x, A)M) \bigcap A$



The *projection* of the point x on the set A is called
$P(x, A) = \varrho(x, A)\mathfrak{B}_1(x) \bigcap A.$



The set of *unit normals* for a set $A \subset E$ at a point $a \in A$ is defined as

$$N^1(a, A) = \{z \in E \mid \exists t > 0 : \; a \in P(a + tz, A), \; \|z\| = 1\}.$$

The set of *unit M-normals* for a set $A \subset E$ at a point $a \in A$ is defined as

$$N^1_M(a, A) = \{z \in E \mid \exists t > 0 : a \in P_M(a + tz, A), \, \mu_M(z) = 1\}.$$

Clark, Stern, Wolenski (1995) and Bernard, Thibault, Zlateva (2006, 2011) considered the following notion in Hilbert and Banach spaces accordingly: a closed set $A \subset E$ is called r-prox-regular if

$$a \in P(a + rz, A) \quad \forall a \in A, \quad \forall z \in N^1(a, A).$$

A closed set $A \subset E$ is called *weakly convex with respect to the quasiball* $M \subset E$ if

$$a \in P_M(a + z, A) \quad \forall a \in A, \quad \forall z \in N^1_M(a, A).$$

$\mathcal{WC}(M)$ denotes the class of weakly convex sets with respect to the quasiball M.

The *convexity modulus* of the quasiball $M \subset E$ is defined as

$$\delta_M(\varepsilon) = \inf \left\{ 1 - \frac{\mu_M(x+y)}{2} \,\middle|\, x, y \in M, \|x - y\| \geq \varepsilon \right\}.$$

A quasiball is called *uniformly convex* if $\delta_M(\varepsilon) > 0$ for any $\varepsilon > 0$.

If the ball $\mathfrak{B}_1(0)$ in the space E is uniformly convex, then the space E is uniformly convex.

The following definition generalizes the notion of uniform convexity:

The set $M \subset E$ is called *boundedly uniformly convex*, if

$$\delta_M(\varepsilon, d) > 0 \quad \forall d > 0 \quad \forall \varepsilon > 0, \quad \text{where}$$

$$\delta_M(\varepsilon, d) = \inf \left\{ 1 - \frac{\mu_M(x+y)}{2} \,\middle|\, x, y \in M \cap \mathfrak{B}_d(0), \|x - y\| \geq \varepsilon \right\}.$$

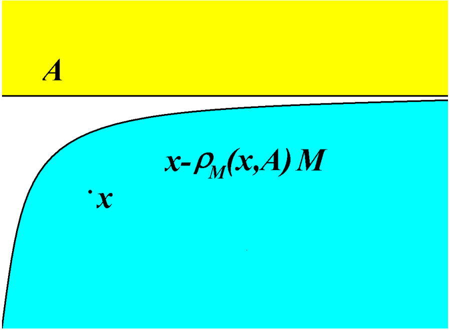
The *smoothness modulus* of the quasiball $M \subset E$ is defined as

$$\beta_M(t) = \left\{ \frac{\mu_M(x+ty) + \mu_M(x-ty)}{2} - 1 \middle| x \in \partial M,\, y \in \partial \mathfrak{B}_1(0) \right\}, \quad t \geq 0.$$

The quasiball is called *uniformly smooth* if $\lim_{t \to +0} \frac{\beta_M(t)}{t} = 0$.

If the ball $\mathfrak{B}_1(0)$ in the space E is uniformly smooth then the space E is uniformly smooth.

Problem: $P_M(x, A) = \emptyset$



The set $M \subset E$ is called *parabolic* if for any $b \in E$ the set $\left(b + \frac{1}{2}M\right) \setminus M$ is bounded.

Note that the epigraph of parabola is parabolic, the epigraph of hyperbola is not parabolic.

A closed convex set M in a Banach space E is called *generating* if for any set X such that the set $A = \bigcap_{x \in X} (M + x)$ is not empty there exists a closed convex set B such that $\overline{A + B} = M$.

Some examples of generating sets:
1) All closed convex sets in $\mathbb{R}^2$
2) The ball in Hilbert space

We call a set C r-strongly convex if $C = \bigcap\limits_{x \in X} \mathfrak{B}_r(x)$.

$\mathcal{SC}(M)$ denotes the class of closed convex sets such that $\exists C_1 : C + C_1 = M$.

If $C \in \mathcal{SC}(\mathfrak{B}_r(0))$ then C is r-strongly convex.

The inverse implication holds if the $\mathfrak{B}_r(0)$ is a generating set in E.

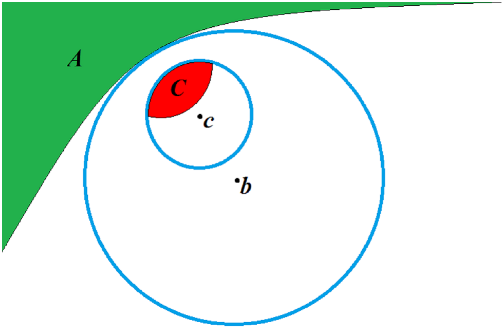
Theorem 1. (M.V. Balashov, G.E. Ivanov)

Let E be a uniformly convex and uniformly smooth Banach space. The following conditions are equivalent:

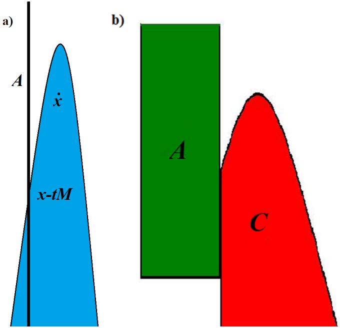
1) for any closed set $A \in \mathcal{WC}(\mathfrak{B}_R(0))$ and a r-strongly convex set C (where $0 < r < R$) there exist points $b, c \in E$ such that

$$C \subset \mathfrak{B}_r(c) \subset \text{int}\, \mathfrak{B}_R(b) \subset E \setminus A.$$

2) the ball in E is a generating set.



Let there be given a quasiball $M \subset E$. The set $A \subset E$ is called M-*closed* if for any $x \in E \setminus A$ the inequality $\varrho_M(x, A) > 0$ holds.



The set $A \subset E$ is called M-*quasibounded*, if it is M-closed and

$$\sup_{\substack{a \in \partial A \\ \|a\| \leq d}} \sup_{z \in N^1_M(a,A)} \|z\| < +\infty, \quad \forall d > 0.$$

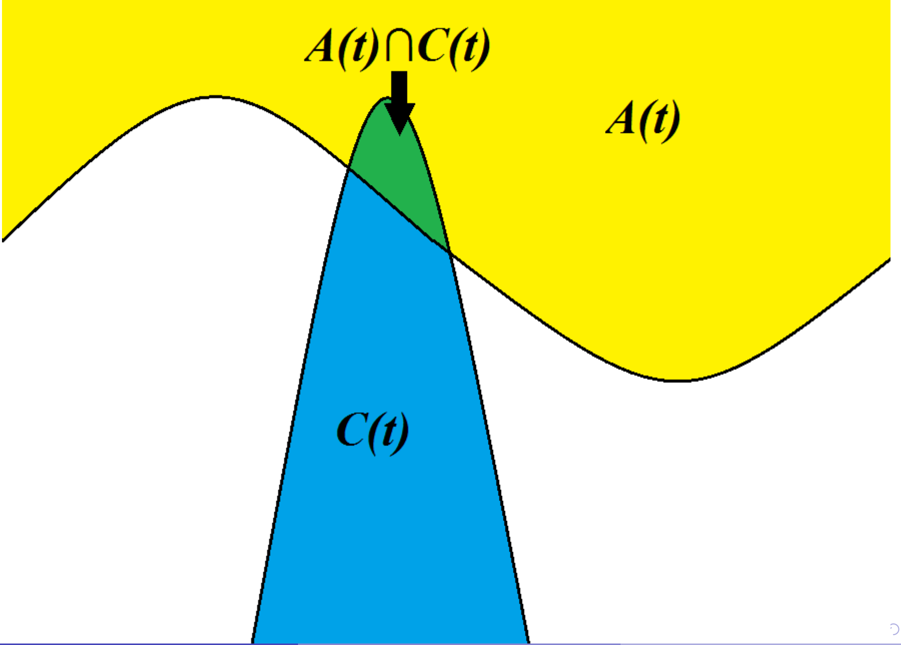
Theorem 2.

Let E be a Banach space and the quasiball $M \subset E$ be parabolic and boundedly uniformly convex. Let $0 < r < R$, the sets $A, C \subset E$ be closed, $A \in \mathcal{WC}(RM)$, $C \in \mathcal{SC}(-rM)$, $A + R\,\mathrm{int}\ M \neq E$. Let at least one of the following statements hold
1) $\varrho_M(C, A) > 0$ *or*
2) $\mathrm{int}\ C \neq \emptyset$, $A \cap \mathrm{int}\ C = \emptyset$ *and the quasiball M is uniformly smooth, the set A is M-quasibounded.*
Then there exist $a, c \in E$ such that $\mathrm{int}\ C \subset c - \mathrm{int}\ rM \subset a - \mathrm{int}\ RM \subset E \setminus A$.

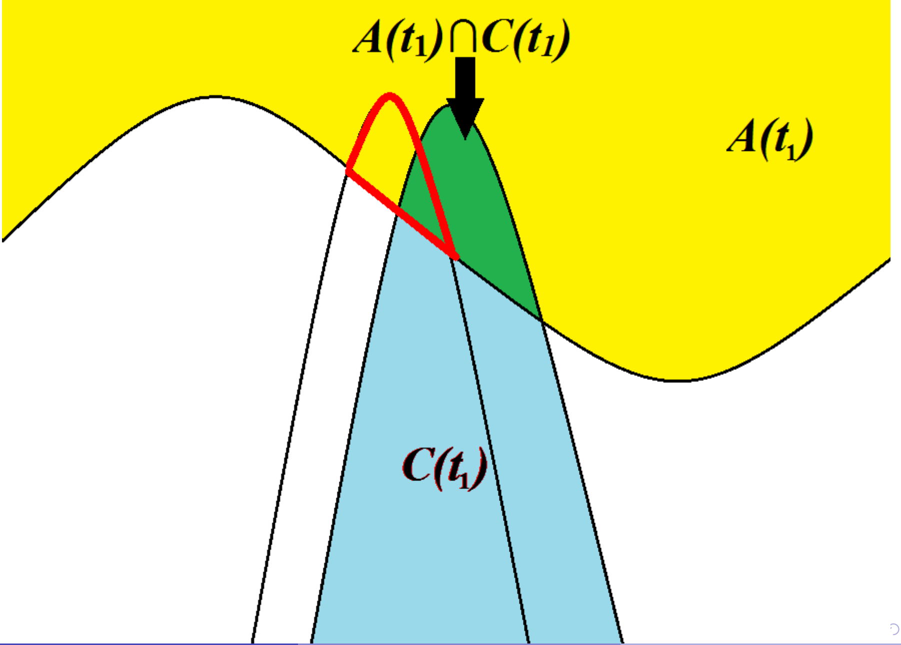
Remark 1.

In case of a finite dimensional space the smoothness condition of E may be aborted.

Applications



Applications



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Thank you for your attention!