Hypomonotonicity of the normal cone and proximal smoothness

G.M. Ivanov

Moscow Institute of Physics and Technology

June 30, 2015

Definition

A set $A \subset X$ is said to be proximally smooth with constant R if the distance function $x \to \rho(x, A)$ is continuously differentiable on set $U(R, A) = \{x \in X : 0 < \rho(x, A) < R\}$.

We denote by $\Omega_{PS}(R)$ the set of all closed proximally smooth sets with constant R in X.

Proposition 1.

Let A be a closed set in a Hilbert space H and $R > 0$. The following conditions are equivalent

1. *the set* $A \in \Omega_{PS}(R)$;
2. *for any vectors* $x_1, x_2 \in A$, $p_1 \in N(x_1, A)$, $p_2 \in N(x_2, A)$ *such that* $\|p_1\| = \|p_2\| = 1$, *the following inequality holds*

$$\langle p_2 - p_1, x_2 - x_1 \rangle \geqslant -\frac{\|x_2 - x_1\|^2}{R}.$$

where

$$N(a_0, A) = \{p \in X^* : \forall \varepsilon > 0 \, \exists \, \delta > 0 : \quad \forall a \in A \cap \mathfrak{B}_\delta(a_0) \\ \langle p, a - a_0 \rangle \leqslant \varepsilon \|a - a_0\|\}.$$

Proposition 1.

Let A be a closed set in a Hilbert space H and $R > 0$. The following conditions are equivalent

1. *the set* $A \in \Omega_{PS}(R)$;
2. *for any vectors* $x_1, x_2 \in A$, $p_1 \in N(x_1, A)$, $p_2 \in N(x_2, A)$ *such that* $\|p_1\| = \|p_2\| = 1$, *the following inequality holds*

$$\langle p_2 - p_1, x_2 - x_1 \rangle \geqslant -\frac{\|x_2 - x_1\|^2}{R}.$$

where

$$N(a_0, A) = \{p \in X^* : \forall \varepsilon > 0 \, \exists \, \delta > 0 : \quad \forall a \in A \cap \mathfrak{B}_\delta(a_0) \\ \langle p, a - a_0 \rangle \leqslant \varepsilon \|a - a_0\| \}.$$

Question

Are the conditions 1) and 2) of Proposition 1 equivalent in an arbitrary Banach space?

SPECIAL BANACH SPACES

Definition

The function $\delta_X(\cdot) : [0, 2] \to [0, 1]$ *is referred to as the modulus of convexity*
$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \ x, y \in \mathfrak{B}_1(o), \ \|x - y\| \geqslant \varepsilon\right\}.$

SPECIAL BANACH SPACES

Definition

The function $\delta_X(\cdot) : [0,2] \to [0,1]$ *is referred to as the modulus of convexity* $\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} :\ x, y \in \mathfrak{B}_1(o),\ \|x-y\| \geqslant \varepsilon\right\}.$

Normed space X is called *unifomly convex*, if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

SPECIAL BANACH SPACES

Definition

The function $\delta_X(\cdot) : [0,2] \to [0,1]$ *is referred to as the modulus of convexity*
$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} :\ x, y \in \mathfrak{B}_1(o),\ \|x-y\| \geqslant \varepsilon\right\}.$

Normed space X is called *unifomly convex*, if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Definition

The function $\rho_X : [0,+\infty) \to \mathbb{R}$ *is referred to as the modulus of smoothness*
$\rho_X(\tau) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 :\ \|x\| = 1, \|y\| = \tau\right\}.$

SPECIAL BANACH SPACES

Definition

The function $\delta_X(\cdot) : [0,2] \to [0,1]$ *is referred to as the modulus of convexity*
$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \ x, y \in \mathfrak{B}_1(o), \ \|x-y\| \geqslant \varepsilon\right\}.$

Normed space X is called *unifomly convex*, if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Definition

The function $\rho_X : [0,+\infty) \to \mathbb{R}$ *is referred to as the modulus of smoothness*
$\rho_X(\tau) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \ \|x\| = 1, \|y\| = \tau\right\}.$

Normed space X is called *uniformly smooth*, if $\lim\limits_{\tau \to +0} \frac{\rho_X(\tau)}{\tau} = 0.$

PREVIOUS RESULTS

Proposition 2.

Let X be a uniformly convex and uniformly smooth Banach space. Let $\rho_X(\tau) \asymp \tau^2$ as $\tau \to 0$. Then the proximally smooth set $A \subset X$ with constant $r > 0$ satisfies condition 2) of Proposition 1 for some constant $R > 0$.

Proposition 3.

Let the convexity and smoothness moduli be of power order at zero in the Banach space X. Let $\delta_X(\varepsilon) \asymp \varepsilon^2$ as $\varepsilon \to 0$. Then, if the set A satisfies condition 2) of Proposition 1, it is proximally smooth with some constant $r > 0$.

Let f and g be two non-negative functions, each one defined on a segment $[0, \varepsilon]$. We shall consider f and g as *equivalent at zero,* denoted by $f(t) \asymp g(t)$ as $t \to 0$, if there exist positive constants a, b, c, d, e such that $af(bt) \leqslant g(t) \leqslant cf(dt)$ for $t \in [0, e]$.

Definition

Let a function $\psi : [0, +\infty) \to [0, +\infty)$ *be given. The set* $A \subset X$ *satisfies the* ψ*-hypomonotonity condition with constant* $R > 0$ *if for some* $\varepsilon > 0$ *and for any* $x_1, x_2 \in A$*,* $p_1 \in N(x_1, A)$*,* $p_2 \in N(x_2, A)$*,* $\|p_1\| = \|p_2\| = 1$ *such that* $\|x_1 - x_2\| \leqslant \varepsilon$*, the inequality*

$$\langle p_2 - p_1, x_2 - x_1 \rangle \geqslant -R\psi\left(\frac{\|x_2 - x_1\|}{R}\right)$$

holds.

Definition

Let a function $\psi : [0, +\infty) \to [0, +\infty)$ *be given. The set* $A \subset X$ *satisfies the* ψ*-hypomonotonity condition with constant* $R > 0$ *if for some* $\varepsilon > 0$ *and for any* $x_1, x_2 \in A$, $p_1 \in N(x_1, A)$, $p_2 \in N(x_2, A)$, $\|p_1\| = \|p_2\| = 1$ *such that* $\|x_1 - x_2\| \leqslant \varepsilon$, *the inequality*

$$\langle p_2 - p_1, x_2 - x_1 \rangle \geqslant -R\psi\left(\frac{\|x_2 - x_1\|}{R}\right)$$

holds.

Through $\Omega_N^{\psi}(R)$ we denote the class of all closed sets $A \subset X$ that satisfy the ψ-hypomonotonity condition with constant $R > 0$.

Definition

Let a function $\psi : [0, +\infty) \to [0, +\infty)$ *be given. The set* $A \subset X$ *satisfies the* ψ*-hypomonotonity condition with constant* $R > 0$ *if for some* $\varepsilon > 0$ *and for any* $x_1, x_2 \in A$, $p_1 \in N(x_1,\, A)$, $p_2 \in N(x_2,\, A)$, $\|p_1\| = \|p_2\| = 1$ *such that* $\|x_1 - x_2\| \leqslant \varepsilon$, *the inequality*

$$\langle p_2 - p_1, x_2 - x_1 \rangle \geqslant -R\psi\left(\frac{\|x_2 - x_1\|}{R}\right)$$

holds.

Through $\Omega_N^{\psi}(R)$ we denote the class of all closed sets $A \subset X$ that satisfy the ψ-hypomonotonity condition with constant $R > 0$.

Definition

Through $\mathfrak{M}$ *denote the class of convex functions* $\psi :\ [0, +\infty) \to [0, +\infty)$ *such that* $\psi(0) = 0$.

Theorem 1.

In a uniformly convex and uniformly smooth Banach space X the following statements are equivalent for the function $\psi \in \mathfrak{M}$:

1. *there exists $k_1 > 0$ such that $\Omega_{PS}(R) \subset \Omega_N^{k_1 \psi}(R)$ for any $R > 0$;*
2. $\rho_X(\tau) = \mathrm{O}(\psi(\tau))$ *as* $\tau \to 0$.

We shall say that function $x(\tau)$ is *big-O* of function $y(\tau)$, and write $x(\tau) = \mathrm{O}(y(\tau))$ as $\tau \to 0$, if the following inequality holds

$$|x(\tau)| \leqslant A|y(\tau)| \quad \forall \tau \in [0, \varepsilon] \quad (\text{for some } \varepsilon > 0, A > 0).$$

Theorem 1.

In a uniformly convex and uniformly smooth Banach space X the following statements are equivalent for the function $\psi \in \mathfrak{M}$:

1. *there exists $k_1 > 0$ such that $\Omega_{PS}(R) \subset \Omega_N^{k_1\psi}(R)$ for any $R > 0$;*
2. *$\rho_X(\tau) = \mathrm{O}(\psi(\tau))$ as $\tau \to 0$.*

We shall say that function $x(\tau)$ is *big-O* of function $y(\tau)$, and write $x(\tau) = \mathrm{O}(y(\tau))$ as $\tau \to 0$, if the following inequality holds

$$|x(\tau)| \leqslant A|y(\tau)| \quad \forall \tau \in [0, \varepsilon] \quad (\text{for some } \varepsilon > 0, A > 0).$$

Lemma 1.

In a uniformly smooth and uniformly convex Banach space X the inclusion

$$(X \setminus \operatorname{int} \mathfrak{B}_1(0)) \in \Omega_N^{\frac{1}{17}\rho_X(\cdot)}(1)$$

holds.

Definition

We say that the function $N : [0, +\infty) \to [0, +\infty)$ *such that* $N(0) = 0$, *satisfies the Figiel condition if there exists a constant* K *such that the function* $N(\cdot)$ *on some interval* $(0, \varepsilon)$ *satisfies the condition*

$$\frac{N(s)}{s^2} \leqslant K \frac{N(t)}{t^2} \quad \forall\, 0 < t \leqslant s < \varepsilon.$$

Definition

We say that the function $N : [0, +\infty) \to [0, +\infty)$ *such that* $N(0) = 0$, *satisfies the Figiel condition if there exists a constant* K *such that the function* $N(\cdot)$ *on some interval* $(0, \varepsilon)$ *satisfies the condition*

$$\frac{N(s)}{s^2} \leqslant K \frac{N(t)}{t^2} \quad \forall\, 0 < t \leqslant s < \varepsilon.$$

Remark 1.

The modulus of smoothness of an arbitrary Banach space satisfies the Figiel condition

Definition

We say that the function $N : [0, +\infty) \to [0, +\infty)$ *such that* $N(0) = 0$, *satisfies the Figiel condition if there exists a constant* K *such that the function* $N(\cdot)$ *on some interval* $(0, \varepsilon)$ *satisfies the condition*

$$\frac{N(s)}{s^2} \leqslant K \frac{N(t)}{t^2} \quad \forall\, 0 < t \leqslant s < \varepsilon.$$

Remark 1.

The modulus of smoothness of an arbitrary Banach space satisfies the Figiel condition

Definition

Through $\mathfrak{M}_2$ *denote the class of functions from* $\mathfrak{M}$ *that satisfy the Figiel condition.*

Theorem 2.

In a uniformly convex and uniformly smooth Banach space X the following statements are equivalent for the function $\psi \in \mathfrak{M}_2$:

1. *there exists $k_2 > 0$ such that $\Omega_N^{k_2 \psi}(R) \subset \Omega_{PS}(R)$ for any $R > 0$;*
2. $\psi(\varepsilon) = \mathrm{O}(\delta_X(\varepsilon))$ *as* $\varepsilon \to 0$.

Theorem 2.

In a uniformly convex and uniformly smooth Banach space X the following statements are equivalent for the function $\psi \in \mathfrak{M}_2$:

1. *there exists $k_2 > 0$ such that $\Omega_N^{k_2\psi}(R) \subset \Omega_{PS}(R)$ for any $R > 0$;*
2. $\psi(\varepsilon) = \mathrm{O}(\delta_X(\varepsilon))$ *as* $\varepsilon \to 0$.

Theorem 3.

Suppose in a Banach space X for some function $\psi \in \mathfrak{M}$ there exist $k_1 > 0$, $k_2 > 0$ such that the inclusions

$$\Omega_N^{k_1\psi}(R) \subset \Omega_{PS}(R) \subset \Omega_N^{k_2\psi}(R)$$

hold. Then $\delta_X(\varepsilon) \asymp \rho_X(\varepsilon) \asymp \varepsilon^2$ as $\varepsilon \to 0$, and, therefore, the space X is isomorphic to a Hilbert space.

OPEN QUESTIONS

Hypothesis 1.

The equality $\Omega_{PS}(R) = \Omega_N^{\psi}(R)$ *holds only in a Hilbert space provided that* $\psi(t) = t^2$.

OPEN QUESTIONS

Hypothesis 1.

The equality $\Omega_{PS}(R) = \Omega_N^{\psi}(R)$ *holds only in a Hilbert space provided that* $\psi(t) = t^2$.

Hypothesis 2.

If in a uniformly convex and uniformly smooth Banach space X *the set* $A \subset X$ *belongs to the class* $\Omega_{PS}(R)$ *and to the class* $\Omega_N^{\psi}(r)$ *for some function* $\psi \in \mathfrak{M} \setminus \mathfrak{M}_2$ *and constants* $R > 0, r > 0$, *then it belongs to the class* $\Omega_N^{\psi_1}(r)$, *where* $\psi_1(t) = ct^2$ *for some* $c \geqslant 0$.

THANK YOU FOR YOUR ATTENTION!