Differential Inclusions with Unbounded Right-hand Side and Optimization Problems

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Pontryagin, L.S., Boltjanskii, V.G, Gamkrelidze, R. V., Mishchenko, E.F. The mathematical theory of control processes. Moscow: Nauka (1969).

Let $\mathcal{R}_T(f, U, X_0, X_1)$ denote the set of all solutions $(x(\cdot), u(\cdot))$ of Cauchy problem for the control nonlinear system

$$x'(t) = f(x(t), u(t)),\ t \in T = [t_0, t_1], u(t) \in U,\ x(t_0) \in X_0, x(t_1) \in X_1.$$

Consider the optimal control problem:

$$\text{Minimize}\ \{\varphi(x(t_1)) \mid (x(\cdot), u(\cdot)) \in \mathcal{R}_T(f, U, X_0, X_1)\}.$$

where $\varphi\colon\ \mathbb{R}^n \to \mathbb{R}^1$ locally Lipschitz function, $X_0, X_1 \subset \mathbb{R}^n$ – the closed sets.

Pontryagin maximum principle

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Pontryagin maximum principle method of proof

Let optimal solution $(\hat{x}(\cdot), \hat{u}(\cdot)) \in \mathcal{R}_T(f, U, X_0, X_1)$ exist.

1) Linearization of system.

$$A(t) = \left(\frac{\partial f_i}{\partial x_j}(\hat{x}(t), \hat{u}(t)) \right), \quad \begin{cases} \overline{x}'(t) = A(t)\overline{x}(t),\ x(t_0) = h, \\ p'(t) = -A^T(t)p(t) \end{cases}$$

– the linear system in variations and the conjugate system.

2) Differentiation by initial data, and continuity $o_\lambda(\cdot)$ of $\overline{x}(\cdot)$.

$\forall \lambda > 0,\ \forall \overline{x}(\cdot)\ \exists x_\lambda(\cdot)\ :\ x_\lambda(\cdot) = \hat{x}(\cdot) + \lambda \overline{x}(\cdot) + o_\lambda(\cdot)$.

3) Local controllability of the linear system implies local controllability of the initial nonlinear system.

$$H(p, x, u) = \langle p, f(x, u) \rangle, \quad x'(t) = \frac{\partial H}{\partial p}, \quad p'(t) = -\frac{\partial H}{\partial x}.$$

4). Necessary conditions of optimality $(\hat{x}(\cdot), \hat{u}(\cdot))$:

a) $\exists\, \hat{p}(\cdot)$ with transversality conditions on the ends t_0, t_1;

b) Pontryagin maximum principle:

$$H(\hat{p}(t), \hat{x}(t), \hat{u}(t)) = \max_{u \in U} H(\hat{p}(t), \hat{x}(t), u),\ \forall t \in T.$$

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Differential inclusion

Controlled system of differential equations was generalized to a differential inclusion:

$$x'(t) \in f(t, x(t), U) \quad \Rightarrow \quad x'(t) \in F(t, x(t)), t \in T.$$

Implicit differential equations, differential inequalities and differential equations with an explosive right-hand side also can be presented as differential inclusions.
see, for example:

Filippov, A.F. The differential equations with an explosive right-hand side. Moscow: Nauka (1985).

Methods for solving optimization problems with differential inclusion

1. Pshenichnyj, B.N., The convex analysis and extreme problems. Moscow: Nauka (1980) (in Russian).

2. Clarke, F.H., Optimization and Nonsmooth Analysis, New York: John Wiley (1983).

3. Morduhovich, B.S., Approximation methods in problems of optimization and control, Moscow: Nauka (1988).

4. Blagodatsky, V.I., Maximum principle for differential inclusions. Proc. of Steklov Inst. of Math., V. 166, 23–43 (1984).

5. Polovinkin, E.S., and G.V.Smirnov. An approach to the differentiation of many-valued mappings, and necessary conditions for optimization... // Diff. Eq. 22, 660-668 (1986).

6. Polovinkin, E.S., and G.V.Smirnov. Time-optimum problem for differential inclusions. Diff. Eq. 22, 940-952 (1986).

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A direct method for solving optimization problems with differential inclusion

G.V.Smirnov and I (1983 – 1995) have created a direct method for study of optimization problems with differential inclusion having bounded measurable -Lipschitz right-hand side in $\mathbb{R}^n$. In short, it consists of the following elements:

a) Pseudo- linearization of differential inclusion;

b) A continuity property of a trajectory set of a differential inclusion;

c) Description of trajectory set of a conjugate convex process (calculation of a polar cone to the trajectory set of convex process);

d) Local controllability of trajectories of differential inclusion;

e) A necessary conditions for solving the initial problem.

This report will outline some generalization of this method for the case of unbounded right-hand side and for Banach spaces.

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Let $T \doteq [t_0, t_1] \subset \mathbb{R}^1$, E —a real separable Banach space, $AC(T, E)$ — the Banach space of absolutely continuous functions $f : T \to E$, with norm $\|f\|_{AC} \doteq \|f(t_0)\|_E + \|f'\|_{L_1}$.

Let us consider $F : T \times E \to \mathcal{P}(E)$ and differential inclusion

$$x'(t) \in F(t, x(t)), \quad t \in T. \tag{1}$$

A function $x(\cdot) \in AC(T, E)$ is called F-trajectory (a solution of the differential inclusion (1)) if its derivative $x' : T \to E$ is a Bochner summable branch of the multivalued mapping $F(\cdot, x(\cdot))$.

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The differential inclusions. Notations and definitions

We denote by $\mathcal{R}_T(F, C_0) \subset AC(T, E)$ a family of all F-trajectories (1) with an arbitrary initial condition $x(t_0) \in C_0 \subset E$.

$AC_\infty(T, E)$ – the subspace of $AC(T, E)$ consisting of all functions f whose derivative $f'(\cdot) \in L_\infty(T, E)$, and the norm is determined by the formula $\|f(\cdot)\|_{AC_\infty} \doteq \|f(t_0)\|_E + \|f'(\cdot)\|_{L_\infty}$.

By $\mathcal{R}_T^\infty(F, C_0)$ we denote the subset of $\mathcal{R}_T(F, C_0)$ consisting of all trajectories $x(\cdot)$ from the space $AC_\infty(T, E)$.

Unbounded differential inclusions

Example of unbounded differential inclusion - controlled differential system with integral restrictions.

The Lipschitz condition for a differential inclusion with unbounded right-hand side as a rule is not fulfilled.

There are various ways of localization of an unbounded multifunction near a point of its graph.
Let X, Y — Banach spaces. Let $\mathcal{P}(Y)$ — the set of all nonempty subsets of Y. Denote $B_\alpha(x_0) \doteq \{x \in X \mid \|x - x_0\| < \alpha\}$.

Let $F : X \to \mathcal{P}(Y)$ – a multifunction, $z_0 \doteq (x_0, y_0) \in \overline{\text{graph} F}$.
An elementary localization is the transition to the mapping

$$G(x) \doteq F(x) \bigcap B_{\alpha_2}(y_0), \quad \forall x \in B_{\alpha_1}(x_0).$$

The multifunction G loses a smoothness of the multifunction F.

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7. Aubin, J.-P., Lipschitz behavior of solutions to convex minimization problems. // Math. Oper. Res. 9, 87Ц111 (1984).

Definition (J.-P. Aubin)

The mapping $F : X \to \mathcal{P}(Y)$ is called a pseudo-Lipschitz continuous with parameter $l > 0$ near a point $z_0 \doteq (x_0, y_0) \in \overline{\text{graph}} F$, if there are numbers $\alpha_1 > 0$, $\alpha_2 > 0$ such that for any $x_1, x_2 \in B_{\alpha_1}(x_0)$ the following inclusion is fair

$$F(x_1) \cap B_{\alpha_2}(y_0) \subset F(x_2) + l \cdot \|x_1 - x_2\| \overline{B_1(0)}.$$

Localization by Ph.D.Loewen & R.T.Rockafellar - 1994, and by A. Ioffe - 2006

8. Loewen, Ph. D. and P.T.Rockafellar. Optimal control of unbounded differential inclusions.// SIAM J. Control and Optimization. V. 32 (2). P. 442–470 (1994).

9. Ioffe, A.D., Existence and relaxation theorems for unbounded differential inclusions.// J. Convex Anal., V. 13 (2). 353–362 (2006).

Let $W \subset T \times \mathbb{R}^n$. We say that F satisfies "the global preudo -Lipschitz condition on W" if $F(t,x) \neq \emptyset$ and there are $\beta \geq 0$ and $l(\cdot) \in L_1(T, \mathbb{R}^1_+)$ such that $\boxed{\text{for all } N > 0}$, $\forall (t, x_1), (t, x_2) \in W$:

$$F(t, x_1) \cap N\overline{B_1(0)} \subset F(t, x_2) + (l(t) + \beta \cdot N)\|x_1 - x_2\|\overline{B_1(0)}.$$

10. Polovinkin, E. S., The existence theorem for solutions of differential inclusion with pseudo-Lipschitz right-hand side, Nelineinyi Mir. 10 (9), P. 571 – 578 (2012).

11. Polovinkin, E.S., Differential inclusions with measurable pseudo-Lipschitz right-hand side.// Proceedings of the Steklov Institute of Math., V. 283, P. 116 – 135 (2013).

12. Polovinkin, E.S., Set-valued analysis and differential inclusions.- Moscow: Fizmatlit, (2014). 520 p.

Let $\mathcal{P}(\mathrm{E})$ denote a set of all nonempty subsets of Banach space $\mathbb{E}$. Let a mapping $\mathrm{F}\colon \mathrm{T}\times\mathrm{E}\to\mathcal{P}(\mathrm{E})$ and a test function $y(\cdot)\in AC(T,E)$ be such that the inequality for distance $\varrho(y'(t),F(t,y(t)))\le\rho(t)$, $\forall t\in T$, be valid, where $\rho(\cdot)\in L_1(T,\mathbb{R}^1_+)$. For example, a test function $y(\cdot)$ may be a F-trajectory $\hat{x}(\cdot)$ with $\rho(\cdot)=0$.

Suppose that there are $\delta > 0$, $l(\cdot), \eta(\cdot) \in L_1(T, \mathbb{R}^1_+)$ and $\xi(\cdot) \in C(T, \mathbb{R}^1_+)$, area W in $T \times E$ such that
1)

$$\xi(t) \geq \delta;\ \rho(t) + l(t)\xi(t) \leq \eta(t),\ \forall t \in T;$$

$$W \supset \{(t, x) \in T \times E \mid \|x - y(t)\| \leq \xi(t),\ t \in T\},$$

2) The set $F(t, x)$ is nonempty and closed for each $(t, x) \in W$.
3) Let for any (t, x_1), $(t, x_2) \in W$ the inclusion be fair

$$F(t, x_1) \cap (y'(t) + \eta(t)B_1(0)) \subset F(t, x_2) + l(t)\|x_1 - x_2\|\overline{B_1(0)}.$$

Definition

Such F we call the measurable–pseudo–Lipschitz in a neighborhood of the test function $y(\cdot)$.

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Such F we call the measurable–pseudo–Lipschitz in a neighborhood of the test function $y(\cdot)$.

Existence theorem

Generalization of Filippov's theorem on the existence (see [10]).

Theorem

Let F: $T \times E \to \mathcal{P}(E)$ be the measurable–pseudo–Lipschitz in a neighborhood of $y(\cdot)$. Then for any $\beta > 0$ and $\delta > 0$ and $x_0 \in B_\delta(y(t_0))$ there is a solution $x(\cdot) \in \mathcal{R}_T(F, x_0)$ differential inclusion (1) with $x(t_0) = x_0$, such that the estimations are fair

$$\|x(t) - y(t)\| \le \xi_\beta(t), \quad \|x'(t) - y'(t)\| \le \eta_\beta(t), \forall t \in T.$$

$$\xi_\beta(t) \doteq e^{m(t)+\beta}\left(\delta + \int_{t_0}^{t} e^{-m(\tau)}\rho(\tau)\,d\tau(1+\beta)\right), \; m(t) \doteq \int_{t_0}^{t} l(\tau)\,d\tau,$$

$$\eta_\beta(t) \doteq l(t)\xi_\beta(t) + \rho(t)(1+\beta), \quad \forall t \in T.$$

Theorem

The received solution continuously depends on perturbations.

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Theorem

The received solution continuously depends on perturbations.

For non-convex sets are well known the lower and the upper tangent (contingent) cones $T_L(A;a)$, $T_U(A;a)$. Also there are well known the convex tangent cones: $T_C(A;a)$, $T_{MP}(A;a)$, $T_{AL}(A;a)$, ...

$$T_C(A;a) \subset T_{MP}(A;a) \subset T_{AL}(A;a) \subset T_L(A;a).$$

$$T_L(A;a) \doteq \liminf_{\lambda \to +0} \lambda^{-1}(A-a) \doteq \{v \in E \mid \lim_{\lambda \to +0} \varrho(v,\ \lambda^{-1}(A-a)) = 0\}.$$

$$T_{AL}(A,a) \doteq T_L(A,a) \overset{*}{-} T_L(A,a),$$

where $A \overset{*}{-} B \doteq \{x \in E \mid x + B \subset A\}$ – Minkowski difference.

Fix the function $\hat{x}(\cdot) \in \mathcal{R}_T(F, X_0)$. For almost every $t \in T$ we consider mapping $F_t(\cdot) \doteq F(t, \cdot)$ and denote the lower derivative $D_L F_t(\hat{x}(t), \hat{x}'(t))(u)$ of the map $F_t(\cdot)$ at the point $(\hat{x}(t), \hat{x}'(t))$ briefly as

$$F'_L(t,u) \doteq \bigcap_{\varepsilon>0} \bigcup_{\delta>0} \bigcap_{\lambda \in (0,\delta)} \left(\lambda^{-1}(F(t, \widehat{x}(t) + \lambda u) - \widehat{x}'(t)) + \overline{B_\varepsilon(0)} \right).$$

We will investigate connections between the trajectory set of differential inclusion in variations

$$u'(t) \in F'_L(t, u(t)), \quad t \in T,\ u(t_0) = u_0,$$

which will be denoted as follows $\mathcal{R}_T(F'_L, u_0)$, and the trajectory set of the initial differential inclusion (1).

The differential inclusion in variations

Fix the function $\hat{x}(\cdot) \in \mathcal{R}_T(F, X_0)$. For almost every $t \in T$ we consider mapping $F_t(\cdot) \doteq F(t, \cdot)$ and denote the lower derivative $D_L F_t(\hat{x}(t), \hat{x}'(t))(u)$ of the map $F_t(\cdot)$ at the point $(\hat{x}(t), \hat{x}'(t))$ briefly as

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We will investigate connections between the trajectory set of differential inclusion in variations

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which will be denoted as follows $\mathcal{R}_T(F'_L, u_0)$, and the trajectory set of the initial differential inclusion (1).

[12]:

Theorem

Let subset $C_0 \subset E$, mapping $F : T \times E \to \mathcal{F}(E)$ and trajectory $\widehat{x}(\cdot) \in \mathcal{R}_T(F, C_0)$ be given. Suppose that F is the measurable–pseudo–Lipschitz in a neighborhood of trajectory $\widehat{x}(\cdot)$. Then for any $u(\cdot) \in \mathcal{R}_T^{\infty}(F'_L, T_L(C_0, \widehat{x}(t_0)))$, there exists such number $\lambda_0 > 0$ that for any $\lambda \in (0, \lambda_0)$ there exist a trajectory $x_\lambda(\cdot) \in \mathcal{R}_T(F, C_0)$ and a function $o(\lambda, \cdot)$ belonging to the space $AC_\infty(T, E)$ such that

$$x_\lambda(t) = \widehat{x}(t) + \lambda u(t) + o(\lambda, t),$$

where

$$\lim_{\lambda \to +0} \lambda^{-1} \|o(\lambda, \cdot)\|_{AC_\infty} = 0.$$

Let us consider the measurable convex process
$Q : T \times E \to \mathcal{P}(E)$:
$Q(t,u) \doteq \{v \in E \mid (u,v) \in K(t)\}$, $t \in T$, $u \in E$, where each set $K(t)$ is a convex closed cone in space $E \times E$ and the map $K : T \to \mathcal{P}(E \times E)$ is measurable. (For example, $Q(t,u) = F'_{AL}(t,u)$). Let $K_0 \subset E$ be a convex closed cone.
Let $\mathcal{R}_T^{\infty}(Q, K_0)$ denote the trajectory set in space $AC_{\infty}(T,E)$ of the Cauchy problem

$$u'(t) \in Q(t, u(t)), t \in T, u(t_0) \in K_0$$

The set $\mathcal{R}_T^{\infty}(Q, K_0)$ is a convex cone in space $AC_{\infty}(T,E)$.

15. Polovinkin, E. S., The computation of the polar cone to the solution set of the differential inclusion // Proceed. of the Steklov Inst. of Math., V. 278, 169 – 178 (2012).

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Let $\mathcal{R}^{\infty}_{T}(Q, K_0)$ denote the trajectory set in space $AC_{\infty}(T,E)$ of the Cauchy problem

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[15]:

Theorem

Let E — separable reflexive Banach space. Let the measurable convex process $Q(t,u) \doteq \{v \in E \mid (u,v) \in K(t)\}$ and the function $\gamma(\cdot) \in L_\infty(T, \mathbb{R}^1_+)$ be such that

$$Q(t,u) \cap \gamma(t)\overline{B_1(0)}\} \neq \emptyset, \ \forall u \in \overline{B_1(0)}, \ t \in T.$$

Then the polar cone $(\mathcal{R}^\infty_T(Q, K_0))^0$ consists of pairs of point $b^* \in E^*$ and function $y^*(\cdot) \in L_1(T, E^*)$ such that for each such pair there is a function $x^*(\cdot) \in L_1(T, E^*)$ for which the following inclusions are valid $(\forall t \in T)$:

$$b^* - \int_{t_0}^{t_1} x^*(s)\,ds \in K_0^0; \quad \left(x^*(t), y^*(t) - \int_t^{t_1} x^*(s)\,ds \right) \in K^0(t),$$

where K_0^0 and $K^0(t)$ – polar cones to cones K_0 and $K(t)$.

Extremal problem on trajectories of a differential inclusion

At the interval $T \doteq [t_0, t_1]$ we consider the problem:

$$\text{Minimize}\,\{\varphi(x(t_1)) \mid x(\cdot) \in \mathcal{R}_T(F, X_0)\}. \tag{2}$$

where $\varphi\colon\ E \to \mathbb{R}^1$ is locally Lipschitz and $X_0 \subset E$ is closed.

Suppose there is a trajectory $\hat{x}(\cdot) \in \mathcal{R}_T(F, X_0)$, such that the mapping $F\colon\ T \times E \to \mathcal{P}(E)$ is measurable – pseudo - Lipschitz in a neighborhood of this trajectory.

Let $K(t) \subset E \times E$ – a closed convex cone, measurable depending on $t \in T$. Suppose we have the following inclusion

$$K(t) \subset T_L(\text{graph}F(t, \cdot); (\hat{x}(t), \hat{x}'(t))) \quad \forall t \in T.$$

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Let $K(t) \subset E \times E$ – a closed convex cone, measurable depending on $t \in T$. Suppose we have the following inclusion

$$K(t) \subset T_L(\mathrm{graph}F(t, \cdot); (\hat{x}(t), \hat{x}'(t))) \quad \forall t \in T.$$

[12]:

Theorem

If $\hat{x}(\cdot)$ is a local solution of Problem (2), then there exists a function $p(\cdot) \in AC(T, E^*)$ satisfying

$$p(t_0) \in T^0_{AL}(X_0, \hat{x}(t_0)), \quad p(t_1) \in \partial^+_{AL}\varphi(\hat{x}(t_1)),$$

$$(p'(t), p(t)) \in K^0(t), \quad \text{a.e. } t \in T. \tag{3}$$

Consequence

The differential inclusion (3) contains a basic condition of the Pontryagin maximum principle, namely, the following equation holds

$$\langle p(t), \hat{x}'(t)\rangle = \max\{\langle p(t), g\rangle \mid g \in F(t, \hat{x}(t))\}, \quad \forall t \in T. \qquad (4)$$

Time-optimum problem

Let $T \doteq [t_0, t_1]$ and $E \doteq \mathbb{R}^n$.

$$\text{Minimize}\,\{t \in (t_0, t_1] \mid x(\cdot) \in \mathcal{R}_T(F, X_0), \quad x(t) \in X_1\}, \qquad (5)$$

where $X_0, X_1 \subset \mathbb{R}^n$ are closed.

Let trajectory $\widehat{x}(\cdot) \in \mathcal{R}_T(F, X_0)$ be a solution of (5). Let $F : T \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be the strictly measurable – pseudo - Lipschitz in a neighborhood of this trajectory.

Let $K(t) \subset \mathbb{R}^n \times \mathbb{R}^n$ be a closed convex cone, measurable depending on $t \in T$. Let

$$K(t) \subset T_L(\text{graph}F(t, \cdot); (\hat{x}(t), \hat{x}'(t))) \quad \forall t \in T.$$

Let K_0 and K_1 be Boltyanskii tents for sets X_0 and X_1 in points $\widehat{x}(t_0)$ and $\widehat{x}(\widehat{t})$ respectively.

Time-optimum problem

Let $T \doteq [t_0, t_1]$ and $E \doteq \mathbb{R}^n$.

$$\text{Minimize}\, \{t \in (t_0, t_1] \mid x(\cdot) \in \mathcal{R}_T(F, X_0), \quad x(t) \in X_1\}, \qquad (5)$$

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Generalized derivative in $\mathrm{AC}(\mathrm{T}, \mathbb{R}^n)$

For any $x(\cdot) \in \mathrm{AC}(\mathrm{T}, \mathbb{R}^n)$ and $t \in (t_0, t_1]$ we define the generalized derivative of $x(\cdot)$ as the set

$$D(x(\cdot), t) \doteq \bigcap_{(\varepsilon > 0)} \bigcap_{(\delta \in (0, t - t_0))} \bigcup_{(\lambda \in (0, \delta))} \left(\frac{x(t) - x(t - \lambda)}{\|x(t - \lambda) - x(t)\| + \lambda} + \overline{B_\varepsilon(0)} \right).$$

The set $D(x(\cdot), t)$ is not empty and bounded. For almost every point $t \in \mathrm{T}$, at which the classical derivative exists, this set is the singleton $\frac{x'(t)}{\|x'(t)\|+1}$.

[12]:

Theorem

Let $\hat{x}(\cdot)$ be a local in $AC(T, \mathbb{R}^n)$ solution of the time-optimum problem (5) and $\widehat{t}$ be the optimal time. Then for any $\xi \in D(\widehat{x}(\cdot), \widehat{t})$ there is nontrivial function $p(\cdot) \in AC(T, \mathbb{R}^n)$ such that

$$p(t_0) \in K_0^0, \quad p(\widehat{t}) \in -K_1^0,$$

$$\langle p(\widehat{t}), \xi \rangle \geq 0,$$

$$(p'(t), p(t)) \in K^0(t), \quad \forall t \in [t_0, \widehat{t}].$$

Thank you for your attention!