Generalized control systems in the space of probability measures

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Introduction

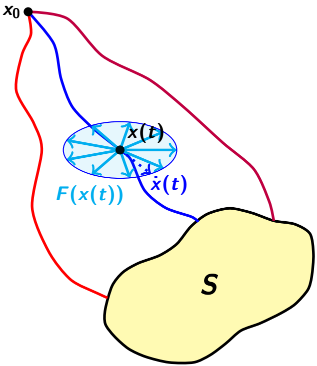
We formulate a Time-Optimal Control Problem in the space of probability measures endowed with the Wasserstein distance as a natural generalization of the classical problem in $\mathbb{R}^d$.

Motivation: to model situations in which we have only a probabilistic knowledge of the initial state.

Classical time-optimal control system

Controlled dynamics:

$$\begin{cases} \dot{x}(t) \in F(x(t)), & \text{for a.e. } t > 0, \\ x(0) = x_0 \in \mathbb{R}^d. \end{cases}$$



Problem: to minimize the time needed to steer x_0 to a given *target set* $S \subseteq \mathbb{R}^d$ not empty, closed.

Hypothesis: $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$, $F(\cdot)$ not empty, convex, compact valued, continuous w.r.t. the Hausdorff metric and with linear growth.

Minimum time needed to steer x_0 to S:

$$T(x_0) := \inf\{\bar{t} > 0 : \exists x(\cdot) \text{ sol. of the control system s.t. } x(\bar{t}) \in S\}.$$

Generalized Problem

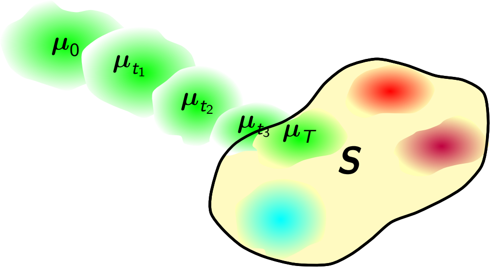
- Initial state: probability measure $\mu_0 \in \mathscr{P}_p(\mathbb{R}^d)$;
- Trajectory: time-depending probability measure on $\mathbb{R}^d$, $\boldsymbol{\mu} := \{\mu_t\}_{t\in[0,T]}$, $\mu_{|t=0} = \mu_0$, (AC curve in $\mathscr{P}_p(\mathbb{R}^d)$);
- Dynamics: since total mass must be preserved during the evolution, the process will be described by a (controlled) continuity equation

$$\partial_t \mu_t + \mathrm{div}(v_t \mu_t) = 0, \qquad \text{for } 0 < t < T;$$

- Control set: v_t to be chosen in the set of $L^2_{\mu_t}-$selections of F (to respect the classical underlying control problem).

Generalized Problem

- Target set: defined by duality (an observer wants to steer the system into states in which the results of some measurements are below a fixed threshold);
- Minimum time: straightforward generalization of the classical one.



State Space

$\mathscr{P}_p(\mathbb{R}^d)$ endowed with the topology induced by the p-Wasserstein distance $W_p(\cdot,\cdot)$, $p \geq 1$.

Let $\mu_1, \mu_2 \in \mathscr{P}_p(\mathbb{R}^d)$, the p-Wasserstein distance is defined as

$$W_p(\mu_1,\mu_2) := \left(\inf \left\{ \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^p \, d\pi \, : \, \pi \in \Pi(\mu_1,\mu_2) \right\} \right)^{1/p}$$

Where the set of admissible transport plans $\Pi(\mu_1,\mu_2)$ is defined by the following

$$\Pi(\mu_1,\mu_2) := \left\{ \pi \in \mathscr{P}_p(\mathbb{R}^d \times \mathbb{R}^d) : \begin{array}{c} \pi(A_1 \times \mathbb{R}^d) = \mu_1(A_1), \\ \pi(\mathbb{R}^d \times A_2) = \mu_2(A_2), \\ \forall A_i \ \mu_i\text{-measurable set }, \\ i = 1,2 \end{array} \right\}$$

Generalized Dynamics

Continuity equation:

$$\begin{cases} \partial_t \mu_t(x) + \mathrm{div}(v_t(x)\mu_t(x)) = 0, & \text{for } 0 < t < T,\ x \in \mathbb{R}^d, \\ \mu_{|t=0} = \mu_0 \in \mathscr{P}_p(\mathbb{R}^d). \end{cases} \tag{1}$$

which represents the conservation of the total mass $\mu_0(\mathbb{R}^d)$.

We require the velocity field $v_t(\cdot)$ to satisfy $v_t(x) \in F(x)\ \forall x \in \mathbb{R}^d$.

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If $v_t(\cdot)$ is locally Lipschitz in x unif. w.r.t. t, we consider the characteristic system:

$$\begin{cases} \dot\gamma(t) = v_t(\gamma(t)), & \text{for a.e. } t \in (0, T) \\ \gamma(0) = x \end{cases}$$

Let $T_t(x)$ denote the unique solution, then: $\mu_t = T_t \sharp \mu_0$, where

$$T_t\sharp\mu_0(B) := \mu_0(T_t^{-1}(B)), \qquad \forall B \subset \mathbb{R}^d,\ B \text{ Borel set}$$

.

Superposition principle: idea

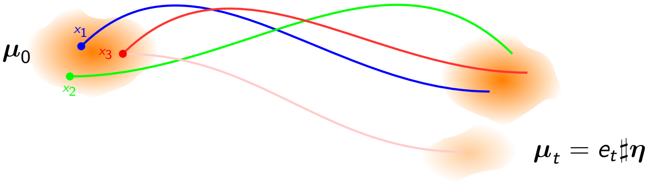
With milder assumptions on v, the (possible not-unique) solution μ_t of the continuity equation can be represented by a superposition of integral solutions of the underlying characteristic system, i.e. of ODEs of the form $\dot{x}(t) = v(x(t))$, or $\dot{x}(t) = v(t, x(t))$.

For this approach, see

L. Ambrosio
The flow associated to weakly differentiable vector fields: recent results and open problems, 2011

and the references therein, where it is also shown that in some cases it is possible to provide conditions on v (assuming for instance Sobolev or BV regularity, and some bounds on the weak derivatives) to recover uniqueness and stability of the solutions in a suitable *smaller class* of measures (Lagrangian flow problem).

Superposition principle: idea



For every point $x \in \operatorname{supp} \mu_0$, consider the set of all integral solutions of $\dot\gamma(t) = v_t \circ \gamma(t)$, $\gamma(0) = x$, and define a probability measure $\boldsymbol{\eta}_x$ on it (if there is a unique solution, $\boldsymbol{\eta}_x$ reduces to a Dirac delta). Let $\boldsymbol{\eta} := \mu_0 \otimes \boldsymbol{\eta}_x$ be the product measure, which is a probability measure on $\mathbb{R}^d \times \Gamma_T$, where $\Gamma_T := C^0([0,T];\mathbb{R}^d)$. For any $\gamma \in \Gamma_T$ consider the evaluation operator $e_t(x,\gamma) = \gamma(t)$. Then $t \mapsto \mu_t = e_t \sharp \boldsymbol{\eta}$ is a solution of the continuity equation. Conversely, every solution can be represented in this way for a suitable $\boldsymbol{\eta}$.

Superposition principle

Let $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]}$ be a solution of the continuity equation $\partial_t\mu_t + \mathrm{div}(v_t\mu_t) = 0$ for a suitable Borel vector field $v : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying

$$\int_0^T \int_{\mathbb{R}^d} \frac{|v_t(x)|}{1+|x|}\, d\mu_t(x)\, dt < +\infty\,.$$

Then there exists a probability measure $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$, with $\Gamma_T = C^0([0, T]; \mathbb{R}^d)$ endowed with the sup norm, such that

(i) $\boldsymbol{\eta}$ is concentrated on the pairs $(x, \gamma) \in \mathbb{R}^d \times \Gamma_T$ such that γ is an absolutely continuous solution of

$$\begin{cases} \dot\gamma(t) = v_t(\gamma(t)), & \text{for } \mathscr{L}^1\text{-a.e } t \in (0, T) \\ \gamma(0) = x, \end{cases}$$

(ii) for all $t \in [0, T]$ and all $\varphi \in C^0_b(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \varphi(x) d\mu_t(x) = \iint_{\mathbb{R}^d\times\Gamma_T} \varphi(\gamma(t))\, d\boldsymbol{\eta}(x, \gamma).$$

Conversely, given any $\boldsymbol{\eta}$ satisfying *(i)* above and defined $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]}$ as in *(ii)* above, we have that $\partial_t\mu_t + \mathrm{div}(v_t\mu_t) = 0$ and $\mu_{|t=0} = e_0\sharp\boldsymbol{\eta}$.

Definition and basic properties of the generalized target

For $p \geq 1$, $\Phi \subseteq C^0(\mathbb{R}^d, \mathbb{R})$ s.t. $\exists x_0 \in \mathbb{R}^d$ with $\phi(x_0) \leq 0$ $\forall \phi \in \Phi$, and for all $\phi \in \Phi$ there exists $D_\phi > 0$ s.t. $\phi(x) \geq -D_\phi$ $\forall x \in \mathbb{R}^d$:

$$\tilde{S}_p^\Phi := \left\{ \mu \in \mathscr{P}_p(\mathbb{R}^d) : \int_{\mathbb{R}^d} \phi(x)\, d\mu(x) \leq 0 \text{ for all } \phi \in \Phi \right\}.$$

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We say that Φ satisfies property (T_p) with $p > 0$ if

(T_p) *for all* $\phi \in \Phi$ *there exist* $A_\phi, C_\phi > 0$ *such that* $\phi(x) \geq A_\phi |x|^p - C_\phi$.

We obtain that:

- $\tilde{S}_p^\Phi$ is closed and convex;
- if (T_p) holds, then $\tilde{S}_p^\Phi$ is compact in the W_p-topology (hence in the w^*-topology).

We say that $\tilde{S}_p^\Phi$ admits a classical counterpart if $\exists S \subseteq \mathbb{R}^d$ s.t.

$$\tilde{S}_p^\Phi = \{ \mu \in \mathscr{P}_p(\mathbb{R}^d) : \operatorname{supp} \mu \subseteq S \}$$

Admissible curves

Let $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$, $\tau > 0$, $\alpha, \beta \in \mathscr{P}(\mathbb{R}^d)$. We say that $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,\tau]} \subseteq \mathscr{P}_p(\mathbb{R}^d)$ is an admissible trajectory defined in $[0,\tau]$ joining α and β, if $\exists \boldsymbol{\nu} = \{\nu_t\}_{t\in[0,\tau]} \subseteq \mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)$ a family of Borel vector-valued measures s.t.

- $J_F(\boldsymbol{\mu}, \boldsymbol{\nu}) < +\infty$, where

$$J_F(\boldsymbol{\mu}, \boldsymbol{\nu}) := \begin{cases} \displaystyle\int_0^\tau \int_{\mathbb{R}^d} \left[1 + I_{F(x)}\left(\frac{\nu_t}{\mu_t}(x)\right)\right] d\mu_t(x)dt, & \text{if } \nu_t \ll \mu_t \text{ for a.e. } t \in [0,\tau], \\ +\infty, & \text{otherwise.} \end{cases}$$

- $\boldsymbol{\mu}$ is a narrowly continuous solution of $\partial_t\mu_t + \mathrm{div}\nu_t = 0$, with $\mu_{t=0} = \alpha$, $\mu_{t=\tau} = \beta$.

In this case, we will shortly say that $\boldsymbol{\mu}$ is driven by $\boldsymbol{\nu}$.

Generalized minimum time

We define the generalized minimum time function $\tilde{T}_p^\Phi : \mathscr{P}_p(\mathbb{R}^d) \to [0, +\infty]$ as:

$$\tilde{T}_p^\Phi(\mu_0) := \inf \left\{ J_F(\boldsymbol{\mu}, \boldsymbol{\nu}) : \begin{array}{l} \boldsymbol{\mu} \text{ is an admissible curve in } [0,\tau] \\ \text{driven by } \boldsymbol{\nu}, \text{ with } \begin{array}{l} \mu_{|t=0} = \mu_0 \\ \mu_{|t=\tau} \in \tilde{S}_p^\Phi \end{array} \end{array} \right\},$$

where, by convention, $\inf \emptyset = +\infty$.

Given $\mu_0 \in \mathscr{P}_p(\mathbb{R}^d)$, an admissible curve $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,\tilde{T}_p^\Phi(\mu_0)]} \subseteq \mathscr{P}_p(\mathbb{R}^d)$, driven by a family of Borel vector-valued measures $\boldsymbol{\nu} = \{\nu_t\}_{t\in[0,\tilde{T}_p^\Phi(\mu_0)]}$, s.t. $\mu_{|t=0} = \mu_0$ and $\mu_{|t=\tilde{T}_p^\Phi(\mu_0)} \in \tilde{S}_p^\Phi$ is optimal for μ_0 if

$$\tilde{T}_p^\Phi(\mu_0) = J_F(\boldsymbol{\mu}, \boldsymbol{\nu}).$$

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Natural questions

- Dynamic programming principle?
- Existence of optimal trajectories?
- Comparison between generalized and classical minimum time? $^{(*)}$
- Controllability results?
- Hamilton-Jacobi-Bellman equation?

$^{(*)}$ We recall that this is possible only if the classical counterpart of the generalized target exists.

Dynamic programming principle

Theorem

Let $0 \leq s \leq \tau$,
$F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be a set-valued function,
$\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,\tau]}$ be an admissible curve for Σ_F.

Then

$$\tilde{T}_p^{\Phi}(\mu_0) \leq s + \tilde{T}_p^{\Phi}(\mu_s).$$

Moreover, if $\tilde{T}_p^{\Phi}(\mu_0) < +\infty$, then

equality holds $\forall s \in [0, \tilde{T}_p^{\Phi}(\mu_0)] \iff \boldsymbol{\mu}$ is optimal for $\mu_0 = \mu_{|t=0}$.

The proof is based on gluing results for solutions of continuity equation.

Compactness of a set of admissible trajectories

Theorem

Assume standard assumptions on F.

Let $\mathscr{A}$ be a set of admissible trajectories defined on $[0, T]$.

Let $C > 0$, $p > 1$ be constants such that for all $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]} \in \mathscr{A}$ it holds $\mathrm{m}_p(\mu_t) \leq C$ for a.e. $t \in [0, T]$.

Then the W_p-closure of $\mathscr{A}$ is a set of admissible trajectories.

Existence theorem

Theorem (Existence of minimizers)

Assume standard assumptions on F.

Let $p > 1$,
$\mu_0 \in \mathscr{P}_p(\mathbb{R}^d)$,
$\Phi \in C^0(\mathbb{R}^d; \mathbb{R})$,
$\tilde{T}_p^{\Phi}(\mu_0) < \infty$.

Then there exists an admissible curve $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]}$ driven by $\boldsymbol{\nu} = \{\nu_t\}_{t\in[0,T]}$ which is optimal for μ_0, that is $\tilde{T}_p^{\Phi}(\mu_0) = J_F(\boldsymbol{\mu}, \boldsymbol{\nu})$.

The proof is based on the previous result of compactness of admissible trajectories in the space of measures, together with the lower semicontinuity of the minimum time functional J_F.

Comparison results (for $\tilde{S}^{\Phi} = \tilde{S}^{\{d_S\}}$)

Proposition

Under the standard assumptions on F we have

$$\tilde{T}_p(\mu_0) \geq ||T||_{L^\infty_{\mu_0}} \qquad \forall \mu_0 \in \mathscr{P}_p(\mathbb{R}^d);$$
$$\tilde{T}_p(\delta_{x_0}) = T(x_0) \qquad \forall x_0 \in \mathbb{R}^d.$$

Theorem

Assume the standard hypothesis on F.

Let $p > 1$,
$\mu_0 \in \mathscr{P}_p(\mathbb{R}^d)$,
$S \subseteq \mathbb{R}^d$ be a weakly invariant set for the dynamics $\dot{x}(t) \in F(x(t))$.

Then

$$\tilde{T}^{\Phi}_p(\mu_0) = \|T(\cdot)\|_{L^\infty_{\mu_0}}.$$

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Controllability in the C_c^1 case

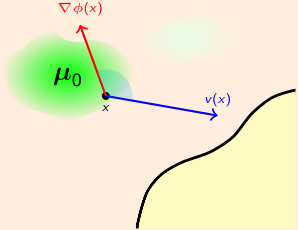
Theorem

Assume the standard hypothesis on F, $p \geq 1$, $\mu_0 \in \mathscr{P}_p(\mathbb{R}^d)$.

Let $\Phi \subseteq C_c^1(\mathbb{R}^d;\mathbb{R})$.

Assume that

$\exists v : \mathbb{R}^d \to \mathbb{R}^d$ Borel vector field,
$\exists \boldsymbol{\mu} := \{\mu_t\}_{t\in[0,+\infty[} \subseteq \mathscr{P}_p(\mathbb{R}^d)$
adm. traj. driven by $\boldsymbol{\nu}$,
with $\boldsymbol{\nu} = \{\nu_t = v\mu_t\}_{t\in[0,+\infty[}$,
$\mu_{|t=0} = \mu_0$,



such that the following controllability condition holds:

(C_c) for all $\phi \in \Phi$ exists $k^\phi > 0$ s.t. $\langle \nabla\phi(x), v(x)\rangle \leq -k^\phi$ for a.e. $t > 0$ and μ_t-a.e. $x \in \mathbb{R}^d$.

Then we have $$\tilde{T}_p^\Phi(\mu_0) \leq \sup_{\phi\in\Phi}\left\{\frac{1}{k^\phi}\int_{\mathbb{R}^d}\phi(x)\,d\mu_0(x)\right\}.$$

Metric gradient: Gangbo & Swiech

Let (X, d) be a complete metric space,
$V : \Omega \subset X \to \mathbb{R}$.

The (upper, lower, bilateral) local slopes of V at x are defined respectively by:

$$|\nabla^{\pm} V(x)| := \limsup_{y \to x} \frac{[V(y) - V(x)]_{\pm}}{d(y,x)}, \quad |\nabla V(x)| := \limsup_{y \to x} \frac{|V(y) - V(x)|}{d(y,x)}.$$

A function $\psi : \Omega \to \mathbb{R}$ is a subsolution test function if $\psi(x) = \psi_1(x) + \psi_2(x)$, where ψ_1, ψ_2 are Lipshitz on every bounded and closed subset of Ω, $|\nabla \psi_1(x)| = |\nabla^- \psi_1(x)|$ is continuous.
A function ψ is a supersolution test function if $-\psi$ is a subsol. test func.

In this way, it is possible to give a definition of viscosity solution for $\mathscr{H}(x, V(x), |DV(x)|) = 0$.

Sub-/Super-differential: Cardaliaguet & Quincampoix

Let $V : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be a continuous function,
$\mu \in \mathscr{P}_2(\mathbb{R}^d)$, $\delta > 0$.

They say that $p_\mu \in L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$ belongs to $D^+_\delta V(\mu)$ if for any $\varphi \in C_b(\mathbb{R}^d;\mathbb{R}^d)$, we have

$$\limsup_{\|\varphi\|_{L^2_\mu}\to 0} \frac{V((Id_{\mathbb{R}^d}+\varphi)\sharp\mu) - V(\mu) - \int_{\mathbb{R}^d}\langle \varphi(x), p_\mu(x)\rangle \, d\mu(x)}{\|\varphi\|_{L^2_\mu}} \leq \delta.$$

Similarly, $q_\mu \in L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$ belongs to $D^-_\delta V(\mu)$ if $-q_\mu \in D^+_\delta[-V](\mu)$.

Sub-/Super-differential & Viscosity solutions in $\mathscr{P}_2(\mathbb{R}^d)$

Let $V : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be a continuous function, $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, $\delta > 0$.

We say that $p_\mu \in L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$ belongs to $D^+_\delta V(\mu)$ if $\forall T > 0$ and $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ s.t. $t \mapsto e_t \sharp \boldsymbol{\eta}$ is an absolutely continuous curve in $\mathscr{P}_2(\mathbb{R}^d)$ defined in $[0, T]$, with $e_0 \sharp \boldsymbol{\eta} = \mu$, we have

$$\limsup_{t \to 0^+} \frac{V(e_t\sharp\boldsymbol{\eta}) - V(e_0\sharp\boldsymbol{\eta}) - \displaystyle\int_{\mathbb{R}^d \times \Gamma_T} \langle p_\mu \circ e_0(x,\gamma), e_t(x,\gamma) - e_0(x,\gamma)\rangle \, d\boldsymbol{\eta}(x,\gamma)}{\|e_t - e_0\|_{L^2_{\boldsymbol{\eta}}}} \le \delta.$$

Similarly, $q_\mu \in L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$ belongs to $D^-_\delta V(\mu)$ if $-q_\mu \in D^+_\delta[-V](\mu)$.

Given $\mathscr{H} : \mathscr{P}_2(\mathbb{R}^d) \times C^0_b(\mathbb{R}^d;\mathbb{R}^d) \to \mathbb{R}$, we say that V is a

1. *viscosity supersolution* of $\mathscr{H}(\mu, DV(\mu)) = 0$ if there exists $C > 0$ s.t. $\mathscr{H}(\mu, q_\mu) \ge -C\delta$ for all $q_\mu \in D^-_\delta V(\mu) \cap C^0_b$.
2. *viscosity subsolution* of $\mathscr{H}(\mu, DV(\mu)) = 0$ if there exists $C > 0$ s.t. $\mathscr{H}(\mu, p_\mu) \le C\delta$ for all $p_\mu \in D^+_\delta V(\mu) \cap C^0_b$.

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$$\limsup_{t \to 0^+} \frac{V(e_t \sharp \boldsymbol{\eta}) - V(e_0 \sharp \boldsymbol{\eta}) - \displaystyle\int_{\mathbb{R}^d \times \Gamma_T} \langle p_\mu \circ e_0(x,\gamma), e_t(x,\gamma) - e_0(x,\gamma) \rangle \, d\boldsymbol{\eta}(x,\gamma)}{\|e_t - e_0\|_{L^2_{\boldsymbol{\eta}}}} \leq \delta.$$

Similarly, $q_\mu \in L^2_\mu(\mathbb{R}^d; \mathbb{R}^d)$ belongs to $D^-_\delta V(\mu)$ if $-q_\mu \in D^+_\delta[-V](\mu)$.

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2. *viscosity subsolution* of $\mathscr{H}(\mu, DV(\mu)) = 0$ if there exists $C > 0$ s.t. $\mathscr{H}(\mu, p_\mu) \leq C\delta$ for all $p_\mu \in D^+_\delta V(\mu) \cap C^0_b$.

Hamiltonian function and Main result

Theorem

Assume standard assumptions on F and that $F(\cdot)$ is bounded.
Then $\tilde{T}_2(\cdot)$ is a viscosity solution of $\mathscr{H}_F(\mu, D\tilde{T}_2(\mu)) = 0$,

where the Hamiltonian function $\mathscr{H}_F : \mathscr{P}_2(\mathbb{R}^d) \times C_b^0(\mathbb{R}^d;\mathbb{R}^d) \to \mathbb{R}$ is defined by $\mathscr{H}_F(\mu,\psi) := -\left[1 + \inf_{\nu\in\mathscr{D}(\mu)} \int_{\mathbb{R}^d} \langle \psi(x), \frac{\nu}{\mu}(x)\rangle \, d\mu\right]$, and

$$\mathscr{D}(\mu) := \left\{ \nu \in \mathscr{M}(\mathbb{R}^d;\mathbb{R}^d) : |\nu| \ll \mu \text{ and } \int_{\mathbb{R}^d} \left(\left|\frac{\nu}{\mu}\right|^2 + I_{F(x)}\left(\frac{\nu}{\mu}(x)\right)\right) d\mu < +\infty \right\}.$$

Sketch of proof : 1) $\tilde{T}_2(\cdot)$ is a subsol. of $\mathscr{H}_F = 0$

- Let $\boldsymbol{\eta} \in \mathscr{T}_F(\mu_0)$, i.e. $\boldsymbol{\eta} \in \mathscr{P}(\mathbb{R}^d \times \Gamma_T)$ concentrated on traj. of $\dot{\gamma}(t) \in F(\gamma(t))$ and s.t. $\gamma(0)\sharp\boldsymbol{\eta} = \mu_0$, and set $\mu_t = e_t\sharp\boldsymbol{\eta}$ for all t;
- by the Dynamic Programming Principle we have $\tilde{T}_2(\mu_0) \leq \tilde{T}_2(\mu_s) + s$ for all $0 < s \leq \tilde{T}_2(\mu_0)$;
- given any $p_{\mu_0} \in D^+_\delta \tilde{T}_2(\mu_0)$, and set

$$A(s, p_{\mu_0}, \boldsymbol{\eta}) := -s - \int_{\mathbb{R}^d \times \Gamma_T} \langle p_{\mu_0} \circ e_0(x,\gamma), e_s(x,\gamma) - e_0(x,\gamma)\rangle \, d\boldsymbol{\eta},$$

$$B(s, p_{\mu_0}, \boldsymbol{\eta}) := \tilde{T}_2(\mu_s) - \tilde{T}_2(\mu_0) - \int_{\mathbb{R}^d \times \Gamma_T} \langle p_{\mu_0} \circ e_0(x,\gamma), e_s(x,\gamma) - e_0(x,\gamma)\rangle d\boldsymbol{\eta},$$

  we have $A(s, p_{\mu_0}, \boldsymbol{\eta}) \leq B(s, p_{\mu_0}, \boldsymbol{\eta})$;
- by definition $p_{\mu_0} \in L^2_{\mu_0} \Rightarrow p_{\mu_0} \circ e_0 \in L^2_{\boldsymbol{\eta}}$;
- we obtain $\left\| \dfrac{e_s - e_0}{s} \right\|_{L^2_{\boldsymbol{\eta}}} \leq C$, and so $\exists \{s_i\}_{i\in\mathbb{N}} \subseteq]0, T[$ such that $s_i \to 0^+$, $\dfrac{e_{s_i} - e_0}{s_i} \rightharpoonup w_{\boldsymbol{\eta}}$ in $L^2_{\boldsymbol{\eta}}$;
- then we divide by $s > 0$, pass to the limit for $s \to 0^+$ and to the infimum on $\boldsymbol{\eta} \in \mathscr{T}_F(\mu_0)$ and we conclude using a result of approximation of L^2_μ−selections of F with C^0_b ones in L^2_μ-norm.

Sketch of proof: 2) $\tilde{T}_2(\cdot)$ is a supsol. of $\mathscr{H}_F = 0$

- Let $\boldsymbol{\eta} \in \mathscr{T}_F(\mu_0)$ and define the admissible trajectory $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]} = \{e_t \sharp \boldsymbol{\eta}\}_{t\in[0,T]}$;
- $q_{\mu_0} \in D^-_\delta \tilde{T}_2(\mu_0)$;
- $\exists \{s_i\}_{i\in\mathbb{N}} \subseteq]0,T[$ such that $s_i \to 0^+$, $\dfrac{e_{s_i} - e_0}{s_i} \rightharpoonup w_{\boldsymbol{\eta}}$ in $L^2_{\boldsymbol{\eta}}$, and $\forall i \in \mathbb{N}$

$$\int_{\mathbb{R}^d\times\Gamma_T} \langle q_{\mu_0} \circ e_0(x,\gamma), \frac{e_{s_i}(x,\gamma) - e_0(x,\gamma)}{s_i} \rangle \, d\boldsymbol{\eta}(x,\gamma)$$

$$\leq 2\delta \left\| \frac{e_{s_i} - e_0}{s_i} \right\|_{L^2_{\boldsymbol{\eta}}} - \frac{\tilde{T}_2(\mu_0) - \tilde{T}_2(\mu_{s_i})}{s_i};$$

- we then take i sufficiently large and argue as in Claim (1) obtaining:

$$\mathscr{H}_F(\mu_0, q_{\mu_0}) \geq -3C\delta + \frac{\tilde{T}_2(\mu_0) - \tilde{T}_2(\mu_{s_i})}{s_i} - 1;$$

- by the D.P.P. $\dfrac{\tilde{T}_2(\mu_0) - \tilde{T}(\mu_s)}{s} - 1 \leq 0$, with equality holdings if and only if $\boldsymbol{\mu}$ is optimal. We conclude by passing to the infimum on all admissible curves.

□

Work in progress

- comparison principle for Hamilton-Jacobi equation;
- Pontryagin maximum principle;
- controllability property with milder assumptions on Φ;
- more general cost functions, possibly considering also density w.r.t. a fixed measure ($\mathscr{L}$), e.g. penalizing concentration;
- application to pedestrian dynamics (evacuation problem).

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Thank you!

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Wasserstein distance

Let $\mu_1, \mu_2 \in \mathscr{P}(\mathbb{R}^d)$, the p-Wasserstein distance is defined as

$$W_p(\mu_1, \mu_2) := \left(\inf \left\{ \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^p \, d\pi \, : \, \pi \in \Pi(\mu_1, \mu_2) \right\} \right)^{1/p}$$

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Where the set of admissible transport plans $\Pi(\mu_1, \mu_2)$ is defined by the following

$$\Pi(\mu_1, \mu_2) := \left\{ \pi \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d) : \begin{array}{c} \pi(A_1 \times \mathbb{R}^d) = \mu_1(A_1), \\ \pi(\mathbb{R}^d \times A_2) = \mu_2(A_2), \\ \forall A_i \ \mu_i\text{-measurable set} \, , \\ i = 1, 2 \end{array} \right\}$$

Wasserstein distance

Let $\mu_1, \mu_2 \in \mathscr{P}(\mathbb{R}^d)$, the p-Wasserstein distance is defined as

$$W_p(\mu_1, \mu_2) := \left(\inf \left\{ \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^p \, d\pi \, : \, \pi \in \Pi(\mu_1, \mu_2) \right\} \right)^{1/p}$$

Let $\{\mu_n\}_{n \in \mathbb{N}} \subseteq \mathscr{P}_p(\mathbb{R}^d)$ and $\mu \in \mathscr{P}_p(\mathbb{R}^d)$ be given, we have

$$\lim_{n \to \infty} W_p(\mu_n, \mu) = 0 \qquad W_p\text{-convergence}$$

iff $\mu_n \rightharpoonup^* \mu$ and $\{\mu_n\}_{n \in \mathbb{N}}$ has unif. integrable p-moments, i.e. iff

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} f(x) \, d\mu_n(x) = \int_{\mathbb{R}^d} f(x) \, d\mu(x),$$

$\forall f : \mathbb{R}^d \to \mathbb{R}$ continuous function s.t. $\exists a, b \geq 0$ and $x_0 \in \mathbb{R}^d$ with $|f(x)| \leq a + b|x - x_0|^p \quad \forall x \in \mathbb{R}^d$.

Generalized distance

For $p \geq 1$, we define the generalized distance

$$\tilde{d}_{\tilde{S}_p^\Phi} : \mathscr{P}(\mathbb{R}^d) \to [0, +\infty[$$
$$\tilde{d}_{\tilde{S}_p^\Phi}(\cdot) := \inf_{\sigma \in \tilde{S}_p^\Phi} W_p(\cdot, \sigma).$$

We obtain that:

- $\tilde{d}_{\tilde{S}_p^\Phi}(\mu) = 0$ if and only if $\mu \in \tilde{S}_p^\Phi$;
- $\tilde{d}_{\tilde{S}_p^\Phi}$ is Lipschitz continuous in $(\mathscr{P}_p(\mathbb{R}^d), W_p)$.

If $\tilde{S}_p^\Phi$ admits a classical counterpart S then $\tilde{S}_p^\Phi = \tilde{S}_p^{\{d_S\}}$ and:

- $\tilde{d}_{\tilde{S}_p^\Phi}(\mu_0) = ||d_S||_{L^p_{\mu_0}}$ with $\mu_0 \in \mathscr{P}_p(\mathbb{R}^d)$.
- $\tilde{d}^p_{\tilde{S}_p^\Phi}$ is convex;
- $\tilde{d}^p_{\tilde{S}_p^\Phi}$ is locally semiconcave w.r. to $W_p^{\min\{p,2\}}$.

Example: non convexity of $\tilde{d}_{\tilde{S}_p}$

Let $p > 1$. In $\mathbb{R}^2$, consider $P = (0,0)$, $Q_1 = (1,0)$, $Q_2 = (0, 2^{1/p})$. Set $S = \{P\}$, hence $\tilde{S}_p := \{\delta_P\}$, and define $\nu_\lambda = \lambda\delta_{Q_1} + (1-\lambda)\delta_{Q_2}$, $\lambda \in [0,1]$. By Monge-Kantorovich duality, we have

$$\begin{aligned} W_p^p(\delta_P, \nu_\lambda) &= \sup_{\substack{\varphi,\psi \in C_b^0(\mathbb{R}^2) \\ \varphi(x)+\psi(y) \le |x-y|^p}} \left\{ \int_{\mathbb{R}^2} \varphi(x)\, d\delta_P(x) + \int_{\mathbb{R}^2} \psi(y)\, d\nu_\lambda(y) \right\} \\ &= \sup \{\varphi(P) + \lambda\psi(Q_1) + (1-\lambda)\psi(Q_2)\} \\ &= \sup \{\lambda(\varphi(P)+\psi(Q_1)) + (1-\lambda)(\varphi(P)+\psi(Q_2))\}. \end{aligned}$$

If we choose $\varphi(P) = 0$, $\psi(Q_1) = 1$, $\psi(Q_2) = 2$, then $\varphi(P) + \psi(Q_i) = |P - Q_i|^p$, $i = 1, 2$. Then we obtain:

$$\tilde{d}^p_{\tilde{S}_p}(\nu_\lambda) = W_p^p(\delta_P, \nu_\lambda) = \lambda + 2(1-\lambda) = 2 - \lambda,$$

whence

$$\tilde{d}_{\tilde{S}_p}(\nu_\lambda) = \sqrt[p]{2-\lambda},$$

which is not convex.

Example: in general $\tilde{T}_p(\mu_0) \not\leq \frac{1}{c}\tilde{d}_{\tilde{S}_p}(\mu_0)$

In $\mathbb{R}^2$, let $S = \{0\}$, $\tilde{S}_p = \tilde{S} = \{\delta_0\}$, $x_0 \in \mathbb{R}^2 \setminus \{0\}$. Define $\mu_0^\lambda := \lambda\delta_0 + (1-\lambda)\delta_{x_0}$, and set $F(x) \equiv \overline{B(0,1)}$ for all $x \in \mathbb{R}^d$. We have that the hypothesis of controllability theorem are satisfied, since S is convex, and by setting $v_t(x) := -x/|x|$ for $x \neq 0$ and $v_t(0) := 0$, we obtain that $\tilde{T}_p(\mu_0^\lambda) = T(x_0)$ for every $\lambda \in [0,1]$. On the other hand,

$$\lim_{\lambda\to 1} W(\mu_0^\lambda, \delta_0) = 0,$$

hence the quotient

$$\frac{\tilde{T}_p(\mu_0^\lambda)}{\tilde{d}_{\tilde{S}_p}(\mu_0^\lambda)}$$

is unbounded as $\lambda \to 1$.

Thank you!