Well-posedness and Subdifferential Calculus of Optimal Value and Infimal Convolution

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Parametrized optimization problem

Let X and P be real Banach spaces,
$h : X \times P \to \mathbb{R} \bigcup \{+\infty\}$ be an objective function

Problem $\mathcal{P}_h$: minimize $h(x, p)$ over $x \in X$ with parameter $p \in P$

Opimal value: $h_{\inf}(p) = \inf\limits_{x \in X} h(x, p), \qquad p \in P$

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We assume that h is lower semicontinuous

Infimal convolution problem

Given lower semicontinuous functions $f, g : X \to \mathbb{R} \bigcup \{+\infty\}$, consider problem

$\mathcal{P}_{f,g}$: $\quad$ minimize $\Big(f(x) + g(p - x)\Big)$ over $x \in X$ with parameter $p \in X$

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Example 1. (*Moreau envelope, Moreau-Yosida regularization*)

$$f_\lambda(p) = \inf_{x \in X} \left(f(x) + \frac{1}{2\lambda} \|p - x\|^2 \right)$$

Infimal convolution problem

Example 2. (the *best approximation problem*)
Given a point $p \in X$ and a set $S \subset X$, the best approximation problem is

$$\text{minimize } \|p - x\| \text{ over } x \in S$$

This is a particular case of the infimal convolution problem $\mathcal{P}_{f,g}$ with $f(x) = \psi_S(x)$ and $g(x) = \|x\|$, where we use $\psi_S(x)$ to denote the *indicator function* of S:

$$\psi_S(x) = \begin{cases} 0, & x \in S, \\ +\infty, & x \notin S \end{cases}$$

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The optimal value of the best approximation problem is distance from the point p to the set S:

$$(f \Box g)(p) = d_S(p) = \inf_{x \in S} \|p - x\|$$

Infimal convolution problem

Example 3. (optimal control problem with constant dynamics) Given a bounded convex set $S \subset X$ with $0 \in \operatorname{int} S$ and a function $f : X \to \mathbb{R} \bigcup \{+\infty\}$, consider the problem

$$\left\{ \begin{array}{l} \text{minimize } t + f(\xi(t,x)) \text{ over } t \geq 0 \text{ and all solutions} \\ \xi(\cdot) = \xi(\cdot,x) \text{ of the differential inclusion } \dot{\xi}(\tau) \in -S \\ \text{a.e. } \tau \in [0,t] \text{ with the initial condition } \xi(0) = x \end{array} \right.$$

the value of this problem is the infimal convolution $f \Box g$ of f and the Minkowski gauge of S:

$$g(x) = \inf\{r \geq 0 : \ x \in rS\}$$

(See recent papers by G.E. Ivanov and L. Thibault [1], [2]).

Well-posedness

The problem $\mathcal{P}_h$ is called *Tykhonov well-posed* at $p_0 \in P$ if it admits a unique solution x_0 and every minimizing sequence for $\mathcal{P}_h$ at p_0 converges to x_0.

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Let $\mathcal{P}_h$ admits at $p_0 \in P$ a unique solution x_0. Define the function $\Delta_{h,p_0} : P \to [0, +\infty)$ as

$$\Delta_{h,p_0}(p) = \inf_{\{x_k\} \text{ is minimizing sequence for } \mathcal{P}_h \text{ at } p} \liminf_{k\to\infty} \|x_k - x_0\|, \qquad p \in P.$$

The problem $\mathcal{P}_h$ is called *approximately well-posed (AWP)* at $p_0 \in P$ if it admits a unique solution at p_0, $h_{\inf}(p)$ is finite for p around p_0 and

$$\lim_{p\to p_0} \Delta_{h,p_0}(p) = 0.$$

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If, in addition, there exists a constant $C > 0$ such that $\Delta_{h,p_0}(p) \le C\|p - p_0\|$ for all p around p_0, then the problem $\mathcal{P}_h$ is called *Lipschitz approximately well-posed (LAWP)* at p_0 with constant C.

Subdifferentials

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Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$ and $\varepsilon \geq 0$, the *Fréchet ε-subdifferential* of f at $x_0 \in \operatorname{dom} f := \{x \in X : \ f(x) \in \mathbb{R}\}$ is

$$\partial^{F,\varepsilon} f(x_0) = \{x^* \in X^* : \ \forall \eta > 0 \ \exists \delta > 0 : \ \forall x \in B_\delta(x_0)$$

$$\langle x^*, x - x_0 \rangle \leq f(x) - f(x_0) + (\varepsilon + \eta)\|x - x_0\|\},$$

where $B_\delta(x_0) = \{x \in X : \ \|x - x_0\| < \delta\}$.
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The *Mordukhovich limiting subdifferential* $\partial^L f(x_0)$ at $x_0 \in \operatorname{dom} f$ is the set of $x^* \in X^*$ such that there exist $\varepsilon_k \downarrow 0$, $x_k \to x_0$ with $f(x_k) \to f(x_0)$, and $x_k^* \to x^*$ weakly star and $x_k^* \in \partial^{F,\varepsilon_k} f(x_k)$ for all $k \in \mathbb{N}$. (see [3]).

Main results

Theorem 1. *Let* $x_0 \in X$ *be a solution of* $\mathcal{P}_h$ *at* $p_0 \in P$*. Then for all* $\varepsilon \geq 0$

$$\{0\} \times \partial^{F,\varepsilon} h_{\inf}(p_0) \subset \partial^{F,\varepsilon} h(x_0, p_0).$$

If, in addition, $\mathcal{P}_h$ *is AWP at* p_0*, then*

$$\{0\} \times \partial^{L} h_{\inf}(p_0) \subset \partial^{L} h(x_0, p_0).$$

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Remark 1. The assumption that $\mathcal{P}_h$ is AWP is essential in Theorem 1.

Remark 2. The reverse inclusions fail in general.
Indeed, consider $h(x,p) = \|x - p\|$, $X = P = \mathbb{R}$. Then $h_{\inf}(p) = 0$ for all $p \in \mathbb{R}$, $\mathcal{P}_h$ is LAWP at $p_0 = 0$ and has a unique solution $x_0 = 0$, but

$$\partial^F h(x_0, p_0) = \partial^L h(x_0, p_0) \not\subset \partial^F h_{\inf}(p_0) = \partial^L h_{\inf}(p_0)$$

Main results

Corollary 1. *Let* $x_0 \in X$ *be a solution of* $\mathcal{P}_{f,g}$ *at* $p_0 \in X$*. Then for all* $\varepsilon \geq 0$

$$\partial^{F,\varepsilon}(f \Box g)(p_0) \subset \Big(\partial^{F,\varepsilon} f(x_0)\Big) \bigcap \Big(\partial^{F,\varepsilon} g(p_0 - x_0)\Big).$$

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Theorem 2. *Suppose that* $x_0 \in X$ *is the solution of* $\mathcal{P}_{f,g}$ *at* $p_0 \in X$ *and the problem* $\mathcal{P}_{f,g}$ *is LAWP at* p_0 *with constant* C*. Then for all* $\varepsilon \geq 0$

$$\Big(\partial^{F,\varepsilon} f(x_0)\Big) \bigcap \Big(\partial^{F,\varepsilon} g(p_0 - x_0)\Big) \subset \partial^{F,(2C+1)\varepsilon}(f \Box g)(p_0).$$

and, consequently,

$$\partial^{F}(f \Box g)(p_0) = \Big(\partial^{F} f(x_0)\Big) \bigcap \Big(\partial^{F} g(p_0 - x_0)\Big).$$

Main results

A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called *lower regular* at a point $x \in \operatorname{dom} f$ (see [3]) whenever $\partial^L f(x) = \partial^F f(x)$.

Corollary 2. *Suppose that* $x_0 \in X$ *is the solution of* $\mathcal{P}_{f,g}$ *at* $p_0 \in X$ *and the problem* $\mathcal{P}_{f,g}$ *is LAWP at* p_0. *Let* f *and* g *be lower regular at points* x_0 *and* $(p_0 - x_0)$ *correspondingly, then* $f \Box g$ *is lower regular at* p_0 *and*

$$\partial^L (f \Box g)(p_0) = \Big(\partial^L f(x_0)\Big) \bigcap \Big(\partial^L g(p_0 - x_0)\Big). \tag{1}$$

Main results

Theorem 3. *Suppose that* $x_0, z_0 \in X$, $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha + \beta > 0$, $f : X \to \mathbb{R} \cup \{+\infty\}$ *and* $g : X \to \mathbb{R}$ *are such that for all* $x, z \in X$

$$f(x) - f(x_0) \geq \alpha \|x - x_0\|,$$

$$g(z) - g(z_0) \geq \beta \|z - z_0\|,$$

$$g(z) - g(z_0) \leq \gamma \|z - z_0\|.$$

Then $\mathcal{P}_{f,g}$ *admits a unique solution* x_0 *at* $p_0 = x_0 + z_0$ *and* $\mathcal{P}_{f,g}$ *is LAWP at* p_0 *with constant* $C = \frac{|\beta| + \gamma}{\alpha + \beta}$.
If, in addition, f *and* g *are lower regular at points* x_0 *and* z_0 *correspondingly, then* $f \square g$ *is lower regular at* p_0 *and the equality (1) holds true.*

Main results

Given a set $S \subset X$ and a point $x_0 \in S$ the *Fréchet normal cone* $N_S^F(x_0)$ and the *Mordukhovich limiting normal cone* $N_S^L(x_0)$ are the Fréchet and Mordukhovich subdifferentials of the indicator function ψ_S at the point x_0.

A set $S \subset X$ is called *normally regular* at $x_0 \in S$ if $N_S^L(x_0) = N_S^F(x_0)$.

Corollary 3. *Suppose that a closed set $S \subset X$ is normally regular at a point $x_0 \in S$. Then the distance function $d_S(\cdot)$ is lower regular at x_0 and*

$$\partial^L d_S(x_0) = N_S^L(x_0) \cap B_X,$$

where B_X is the unit ball of X.

Bibliography

[1] G.E. Ivanov and L. Thibault, Infimal convolution and optimal time control problem I: Fréchet and proximal subdifferentials, Set-Valued and Variational Analysis, doi:10.1007/s11228-016-0398-z (2017)

[2] G.E. Ivanov and L. Thibault, Infimal convolution and optimal time control problem II: Limiting subdifferential, Set-Valued and Variational Analysis, doi: 10.1007/s11228-017-0402-2 (2017)

[3] B. S. Mordukhovich, Approximation Methods in Problems of Optimization and Control, John Wiley & Sons, New York (2005).

[4] L. Thibault, On subdifferentials of optimal value functions, SIAM J. Control Optim. 29 (5) (1991), 1019-1036.

[5] H. Van Ngai, D. The Luc, M. Théra, Extensions of Fréchet ε-subdifferential calculus and applications, J. Math. Anal. Appl. 268 (2002), 266-290.

[6] H. Van Ngai, J.-P. Penot, Subdifferentiation of regularized functions, Set-Valued and Variational Analysis, Springer, 24, (2016), 167-189.

Thank you!