Weak convexity of functions and the infimal convolution

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Inf- and sup- convolutions

Let E be a real Banach space

The *infimal convolution* of functions $f, g : E \to \mathbb{R} \cup \{+\infty\}$ is

$$(f \boxplus g)(x) = \inf_{x_1,\, x_2 \in E:\ x_1 + x_2 = x} \Big(f(x_1) + g(x_2) \Big), \qquad x \in E.$$

The *sup-convolution* of f and g is

$$(f \boxminus g)(x) = \sup_{x_1 \in E,\, x_2 \in \mathrm{dom}\ g:\ x_1 - x_2 = x} \Big(f(x_1) - g(x_2) \Big), \qquad x \in E.$$

Any of these operations may be expressed through another one and the operation

$$f_-(x) = -f(-x), \qquad x \in E$$

$$(f \boxplus \gamma)_- = f_- \boxminus \gamma, \qquad (f \boxminus \gamma)_- = f_- \boxplus \gamma.$$

Φ-convexity

For a family Φ of functions $\varphi : E \to \mathbb{R}$, a function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be Φ-*convex*, if there exists a subset $\Phi' \subset \Phi$ such that

$$f(x) = \sup_{\varphi \in \Phi'} \varphi(x), \qquad x \in E.$$

This concept was considered in

S. Dolecki, S. Kurcyusz: On Φ-convexity in extremal problems, SIAM J. Control Optim. 16 (1978) 277–300.

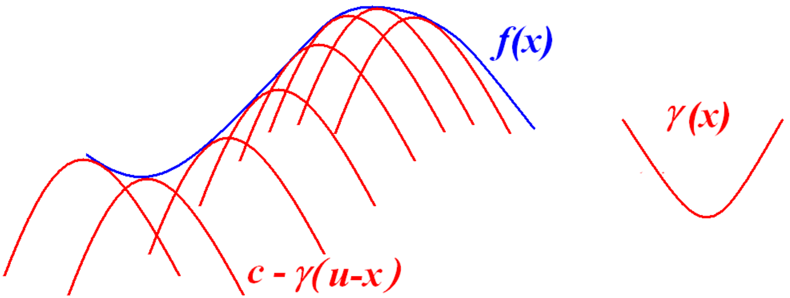
J.E. Martinez-Legaz, I. Singer: On Φ-convexity of convex functions, Linear Algebra and Its Applications, 278 (1998) 163–181.

A. M. Rubinov: Abstract convexity and global optimization, Kluwer Academic Publishers, Dordrecht (2000).

Class $\mathcal{CWC}(\gamma)$

Given a function $\gamma : E \to \mathbb{R}$, a function $f : E \to \mathbb{R} \cup \{+\infty\}$ is said to be *convolutionally weakly convex* with respect to $\gamma : E \to \mathbb{R}$ (we write $f \in \mathcal{CWC}(\gamma)$) if f is Φ-convex with family

$$\Phi = \Phi(\gamma) = \{\varphi : E \to \mathbb{R} \mid \exists u \in E,\ c \in \mathbb{R} :\ \varphi(x) = c - \gamma(u - x),\ \forall x \in E\}.$$



Class $\mathcal{CWC}(\gamma)$

Lemma 1

For any functions $\gamma : E \to \mathbb{R}$ *and* $f : E \to \mathbb{R} \cup \{+\infty\}$ *with* $\operatorname{dom} f \neq \emptyset$ *the following assertions are equivalent:*
(i) $f \in \mathcal{CWC}(\gamma)$*;*
(ii) there exists a function $g : E \to \mathbb{R} \cup \{+\infty\}$ *such that* $f = g \boxminus \gamma$*;*
(iii) $f = f \boxplus \gamma \boxminus \gamma$*.*

Consequently $\mathcal{CWC}(\gamma)$ is the same as the class of exactly γ-regular functions in the sense of
J.-E. Martinez-Legaz, J.-P. Penot: Regularization by erasement. Math. Scand. 98 (2006) 97–124.

Moreau–Yosida regularization

Let H be a real Hilbert space.
For $t > 0$ and function $f : H \to \mathbb{R} \cup \{+\infty\}$, the *Moreau–Yosida regularization* (or the *Moreau envelope*, or the *proximal envelope*) is

$$f^t(x) = \inf_{u \in H} \Big(f(x) + \frac{1}{2t} \|x - u\|^2 \Big), \qquad x \in H.$$

This can be rewritten in terms of the infimal convolution as $f^t = f \boxplus \beta_t$ with

$$\beta_t(x) = \frac{1}{2t} \|x\|^2, \qquad t > 0, \quad x \in H.$$

Properties.
1. If f is lower semicontinuous at $x_0 \in \mathrm{dom}\, f$, then $f^t(x_0) \to f(x_0)$ as $t \to +0$.
2. The Moreau–Yosida regularization preserves the minimizers of the function.
3. If $T > 0$ and $f \in \mathcal{CWC}(\beta_T)$ (in particular, if f is convex), then for any $t \in (0, T)$ the function f^t is smooth, i.e. it possesses a Lipschitz continuous derivative.

Lasry–Lions regularization

If $f \notin \mathcal{CWC}(\beta_T)$ for all $T > 0$, instead of the Moreau–Yosida regularization one can use the Lasry–Lions regularization

$$f^{t,s} = f \boxminus \beta_t \boxplus \beta_s$$

which is also smooth and converges to f as $0 < s < t \to +0$.

These properties give a lot of applications of the Moreau–Yosida and Lasry–Lions regularizations in Hamilton–Jacobi equations and optimization.

Calculus of convexity parameters and smoothness

Lemma 2

If $E = H$ is a Hilbert space, $f \in \mathcal{CWC}(\beta_t)$, $0 < s < t$, then $f \boxplus \beta_s \in \mathcal{CWC}(\beta_{t-s})$.

Theorem 1

Let $E = H$ be a Hilbert space, $t > 0$. Then for a function $f : E \to \mathbb{R}$ the following conditions are equivalent:
(a) $f \in \mathcal{CWC}(\beta_t)$ and $f_- \in \mathcal{CWC}(\beta_t)$
(b) f is Fréchet differentiable and the derivative f' is Lipschitz continuous with constant $\frac{1}{t}$ on H.

Proof of property 3. If $f \in \mathcal{CWC}(\beta_T)$, $t \in (0, T)$, then by Lemma 2 we get $f \boxplus \beta_t \in \mathcal{CWC}(\beta_{T-t})$. On the other hand, Lemma 1 implies that $(f \boxplus \beta_t)_- = f_- \boxminus \beta_t \in \mathcal{CWC}(\beta_t)$. Hence, by Theorem 1 we deduce that the derivative of the Moreau–Yosida regularization $f^t = f \boxplus \beta_t$ does exist and is Lipschitz continuous with constant $\max\left\{\frac{1}{T-t}, \frac{1}{t}\right\}$ on H.

Class $\mathcal{RCWC}(\gamma)$

Given a function $\gamma : E \to \mathbb{R} \cup \{+\infty\}$ and a number $t > 0$, we denote

$$\gamma_t(x) = t \cdot \gamma\left(\frac{x}{t}\right), \qquad x \in E. \tag{1}$$

A function $f \in \mathcal{CWC}(\gamma)$ is called *regularly convolutionally weakly convex* w.r.t. γ (we write $f \in \mathcal{RCWC}(\gamma)$) if

$$\forall \varepsilon \in (0,1)\ \exists \delta \in (0,1):\ \forall t \in (0,\delta) \qquad f \boxplus \gamma_t \in \mathcal{CWC}(\gamma_{1-\varepsilon}).$$

Lemma 2 implies that the classes $\mathcal{RCWC}(\beta_t)$ and $\mathcal{CWC}(\beta_t)$ coincide provided that the space is Hilbert and $\beta(x) = \|x\|^2$.

In general $\mathcal{RCWC}(\gamma) \neq \mathcal{CWC}(\gamma)$. For example in case $E = \mathbb{R}^2$ with $\gamma(x_1, x_2) = (x_1^2 + x_2^2)^2$.

Class $\mathcal{WC}(M)$

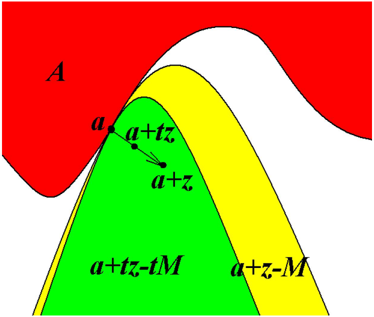
A set $M \subset E$ is called a *quasiball* if it is closed convex and $0 \in \operatorname{int} M$.

$$\mu_M(x) = \inf\{t > 0 : \ x \in tM\} \qquad - \quad \text{Minkowski functional of } M,$$

$$P_M(x, A) = \operatorname*{argmin}_{a \in A} \mu_M(x - a) \qquad - \quad \text{metric projection w.r.t. } M.$$

A set $A \subset E$ is called *weakly convex* with respect to a quasiball M if

$$a \in P_M(a + tz, A), \ t > 0 \quad \Rightarrow \quad a \in P_M(a + z, A),$$



Main result

Theorem 2

Let a function $\gamma : E \to \mathbb{R}$ *be coercive (i.e.* $\lim\limits_{\|x\|\to+\infty} \frac{\gamma(x)}{\|x\|} = +\infty$*), bounded on any bounded set and uniformly convex on any convex bounded set. Then*

$$\mathcal{RCWC}(\gamma) = \mathcal{WC}(\text{epi } \gamma).$$

Thank you!