# Properties of weakly convex sets in asymmetric normed spaces

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Let $E$ be a real Banach space.

A set $M$ is called quasiball if $M$ is convex closed and $0 \in \operatorname{int} M$.

The *ball* of radius $R$ and center $a$ is $\mathfrak{B}_R(a) = \{x \in E : \|x - a\| \le R\}$.

The *Minkowski functional* of the quasiball $M$

$\mu_M(x) = \inf\left\{t > 0 \middle|\, x \in tM\right\}$

is the nonsymmetric seminorm.

The Minkowski functional of the unit ball $\mu_{\mathfrak{B}_1(0)}(x) = \|x\|$.

The $M$-*distance* from a point $x \in E$ to the set $A \subset E$ is

$\varrho_M(x, A) = \inf\limits_{a \in A} \mu_M(x - a) =$

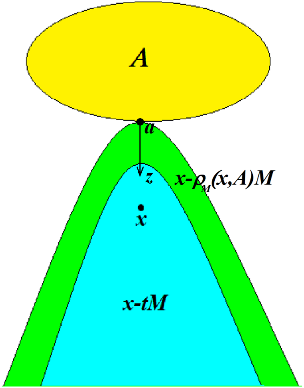
$= \inf\{t > 0 \,|\, (x - tM) \bigcap A \ne \emptyset\}$.

The *distance* from the point $x \in E$ to the set $A \subset E$ is

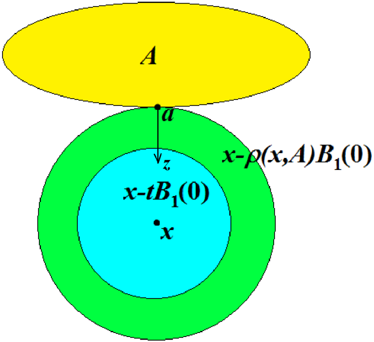
$\varrho(x, A) = \inf\limits_{a \in A} \|x - a\| =$

$= \inf\{t > 0 \,|\, (x - \mathfrak{B}_t(0)) \bigcap A \ne \emptyset\}$.

The *M-projection* of the point $x \in E$ on the set $A \subset E$ is called
$P_M(x, A) = (x - \varrho_M(x, A)M) \bigcap A$




The *projection* of the point $x$ on the set $A$ is called
$P(x, A) = \varrho(x, A)\mathfrak{B}_1(x) \bigcap A.$

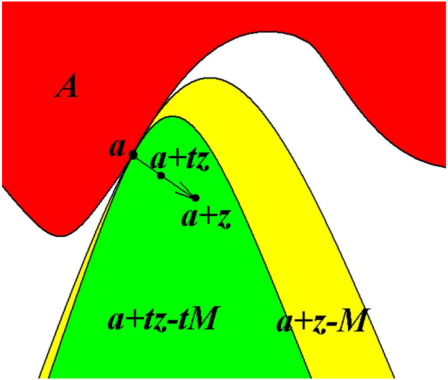


The set of *unit normals* for a set $A \subset E$ at a point $a \in A$ is defined as

$$N^1(a, A) = \{z \in E \mid \exists t > 0: \ a \in P(a + tz, A), \ \|z\| = 1\}.$$

The set of *unit M-normals* for a set $A \subset E$ at a point $a \in A$ is defined as

$$N^1_M(a, A) = \{z \in E \mid \exists t > 0: \ a \in P_M(a + tz, A), \ \mu_M(z) = 1\}.$$



Clark, Stern, Wolenski (1995) and Bernard, Thibault, Zlateva (2006, 2011) considered the following notion in Hilbert and Banach spaces accordingly: a closed set $A \subset E$ is called $r$-prox-regular if

$$a \in P(a + rz, A) \quad \forall a \in A, \quad \forall z \in N^1(a, A).$$

A closed set $A \subset E$ is called *weakly convex with respect to the quasiball* $M \subset E$ if

$$a \in P_M(a + z, A) \quad \forall a \in A, \quad \forall z \in N^1_M(a, A).$$

$\mathcal{WC}(M)$ denotes the class of weakly convex sets with respect to the quasiball $M$.

The *convexity modulus* of a set $A$ is defined as

$$\delta(\varepsilon, A) = \sup\left\{\delta > 0 \,\middle|\, B_\delta\left(\frac{x+y}{2}\right) \subset A,\ \forall x, y \in A:\ \|x-y\| \geq \varepsilon\right\}.$$

A set $A$ is called *uniformly convex* if $\delta(\varepsilon) > 0$ for any $\varepsilon > 0$.

If the ball $\mathfrak{B}_1(0)$ in the space $E$ is uniformly convex, then the space $E$ is uniformly convex.

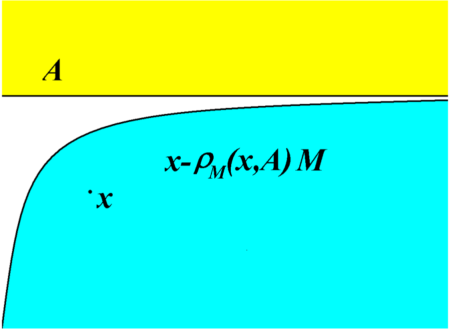
The following definition generalizes the notion of uniform convexity:

The set $M \subset E$ is called *boundedly uniformly convex*, if

$$\delta_M(\varepsilon, d) > 0 \quad \forall d > 0 \quad \forall \varepsilon > 0, \quad \text{where}$$

$$\delta_M(\varepsilon, d) = \inf\left\{1 - \frac{\mu_M(x+y)}{2} \,\middle|\, x, y \in M \cap B_d(0), \|x-y\| \geq \varepsilon\right\}.$$

Problem: $P_M(x, A) = \emptyset$



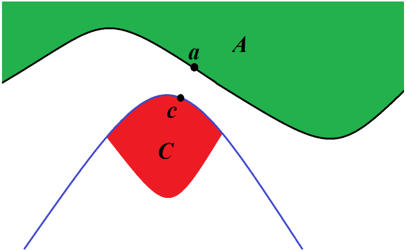

The set $M \subset E$ is called *parabolic* if for any $b \in E$ the set $\left(b + \frac{1}{2}M\right) \setminus M$ is bounded.

Note that the epigraph of parabola is parabolic, the epigraph of hyperbola is not parabolic.

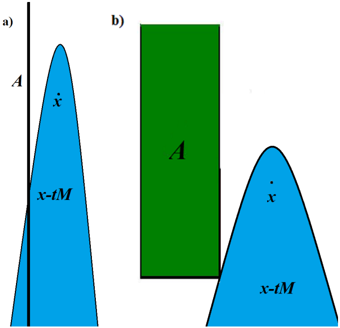
A set $C$ is called *strongly convex* with respect to the quasiball $M$ if there exists $C_1 \subset E$ such that $C + C_1 = M$.
$\mathcal{SC}(M)$ denotes the class of strongly convex sets with respect to the quasiball $M$.

**Theorem (Ivanov, Lopushanski 2015.)**

*Let the set $C \subset E$ be such that $C \in \mathcal{SC}(-rM)$, and the set $A \in \mathcal{WC}(RM)$, where $0 < r < R$ and let both of the sets be closed. Let $0 < \varrho_M(C, A) < R - r$. Then the problem $\min\limits_{c\in C,\ a\in A} \mu_M(c - a)$ is well-posed.*



Let there be given a quasiball $M \subset E$. The set $A \subset E$ is called $M$-*closed* if for any $x \in E \setminus A$ the inequality $\varrho_M(x, A) > 0$ holds.




The set $A \subset E$ is called $M$-*quasibounded*, if it is $M$-closed and

$$\sup_{\substack{a \in \partial A \\ \|a\| \leq d}} \sup_{z \in N_M^1(a,A)} \|z\| < +\infty, \quad \forall d > 0.$$

Note that if $M$ is the epigraph of a convex continuous coercive function and the set $A$ is the epigraph of a Lipschitz function, then the set $A$ is $M$-quasibounded.

(We say that a function $f : E \to \mathbb{R}$ is *coercive* if lim $\lim\limits_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = +\infty$.)

### Theorem (Ivanov, Lopushanski 2015.)

*Let the quasiball $M$ in a Banach space $E$ be parabolic and boundedly uniformly convex. Let the set $A \in \mathcal{WC}(M)$ be $M$-quasibounded. Let $C \in \mathcal{SC}(-rM)$, where $r \in (0,1)$ be closed and* $\operatorname{int} C \neq \emptyset$*. Let $\varrho_M(C, A) < 1 - r$, $A \bigcap \operatorname{int} C = \emptyset$. Then the problem $\min\limits_{a\in A,\ c\in C} \mu_M(c - a)$ is well posed.*

The *smoothness modulus* of the ball $\mathfrak{B}_1(0) \subset E$ is defined as

$$\beta(t) = \left\{ \frac{\|x + ty\| + \|x - ty\|}{2} - 1 : \ \|x\| = \|y\| = 1 \right\}, \quad t \geq 0.$$

The quasiball is called *uniformly smooth* if $\lim_{t\to+0} \frac{\beta(t)}{t} = 0$.

If the ball $\mathfrak{B}_1(0)$ in the space $E$ is uniformly smooth then the space $E$ is uniformly smooth.

The quasiball $M \subset E$ is called *boundedly uniformly smooth*, if

$$\lim_{t\to+0} \frac{\beta_M(t, R)}{t} = 0 \qquad \forall R > \sigma_M,$$

where $\sigma_M$ is such that $\mathfrak{B}_{\sigma_M}(0) \subset M$ and for any $t \geq 0$ and $R > \sigma_M$

$$\beta_M(t, R) = \sup \left\{ \frac{\mu_M(x + ty) + \mu_M(x - ty)}{2} - 1 : \ x \in \partial M \cap \mathfrak{B}_R, \ y \in \mathfrak{B}_1 \right\}.$$

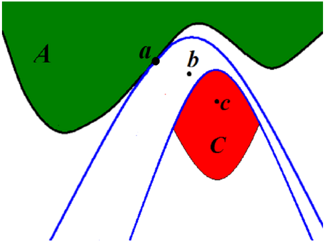
## Theorem 1.

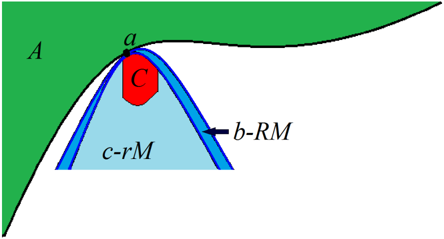
*Let $E$ be a Banach space and the quasiball $M \subset E$ be parabolic and boundedly uniformly convex. Let $0 < r < 1$, the sets $A, C \subset E$ be closed, $A \in \mathcal{WC}(M)$, $C \in \mathcal{SC}(-rM)$, $A + \operatorname{int} M \neq E$. Let at least one of the following statements hold*

*1)* $\varrho_M(C, A) > 0$ *or*

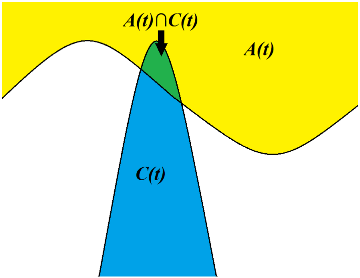
*2)* $\operatorname{int} C \neq \emptyset$, $A \cap \operatorname{int} C = \emptyset$ *and the quasiball $M$ is boundedly uniformly smooth, the set $A$ is $M$-quasibounded.*

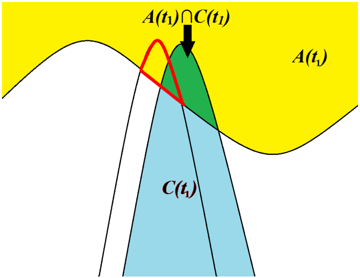
*Then there exist $b, c \in E$ such that* $\operatorname{int} C \subset c - \operatorname{int} rM \subset b - \operatorname{int} M \subset E \setminus A$.



# Applications



The *Fréchet normal cone* to the set $A$ at $x \in A$ is

$$N^F(x,A) = \{\xi \in E^* | \quad \forall \gamma > 0 \quad \exists \delta > 0 : \\ \langle \xi, a - x \rangle \leq \gamma \, \|a - x\|, \quad \forall a \in \mathfrak{B}_\delta(x) \bigcap A\}.$$

The *support function* of the set $M \subset E$ is

$$s(p,M) = \sup_{x \in M} \langle p, x \rangle, \quad p \in E^*.$$

Given a functional $p \in E^*$, if $p \in b(M) \setminus \{0\}$ we define

$$J_M^*(p) = \{x \in E : \langle p, x \rangle = (p,M)\mu_M(x), s(p,M) = \mu_M(x)\},$$

otherwise we put $J_M^*(p) = \{0\}$.

The *proximal M-normal cone* to the set $A$ at a point $a \in \partial A$ is

$$N_M^P(a, A) = \{p \in b(M) \,|\, \exists z \in J_M^*(p),\ \exists t > 0:\ a \in P_M(a + tz, A)\}.$$

The *Mordukhovich limiting cone* is

$$N_M^L(x, A) = {}^{w^*-seq}\limsup_{y \to x} N_M^P(y, A) = \{{}^{w^*} - \lim_{n\to\infty} x_n^* : \\ x_n^* \in N_M^P(x_n, A), x_n \in A, x_n \to x, n \to \infty\},$$

where ${}^{w^*} - \lim$ means the limit with respect to weak* topology.

## Theorem 2.

*Let $E$ be a reflexive Banach space. Let the quasiball $M$ be boundedly uniformly smooth, boundedly uniformly convex and parabolic. Let the set $A \in \mathcal{WC}(M)$ be $M$-quasibounded. Then*

$$N^F(x, A) = N_M^P(x, A) = N_M^L(x, A), \quad \forall x \in A.$$

- F. H. Clarke , R. J. Stern, P. R. Wolenski, *Proximal Smoothness and Lower–$C^2$ Property*, J. Convex Analisys, 2, No. 1,2, 117–144, 1995.
- R. A. Poliquin, R. T. Rockafellar, *Prox-regular functions in variational analysis*, Trans. Amer. Math. Soc., 348 1805–1838, 1996.
- F. Bernard, L. Thibault and N. Zlateva, *Characterizations of prox-regular sets in uniformly convex Banach spaces*, J. Convex Analysis, 13, 525–559, 2006.
- G. E. Ivanov, M. S. Lopushanski, *Well-Posedness of Approximation and Optimization Problems for Weakly Convex Sets and Functions*, J. Mathematical Sciences, 209, No. 1, 66–87, 2015.
- G. E. Ivanov, *Continuity and selections of the intersection operator applied to nonconvex sets*, J. Convex Analysis, 22, No. 4, 932–962, 2015.
- G. E. Ivanov, *Weak Convexity of Sets and Functions in a Banach Space*, J. Convex Analysis, 22, No. 2, 365–398, 2015.

- G. E. Ivanov, *Weak Convexity of Functions and the Infimal Convolution*, J. of Convex Analysis, 23, 719–732, 2016.
- G. E. Ivanov, M. S. Lopushanski, *Separation theorems for nonconvex sets in spaces with non-symmetric seminorm*, J. Math. Ineq. and Applications, accepted in 2016.
- M. S. Lopushanski, *Normal Regularity of Weakly Convex Sets in Asymmetric Normed Spaces*, J. of Convex Analysis, accepted in 2017.
- B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation II. Applications*, Springer, Berlin (2006).

Thank you for your attention!