Gromov's waist of cubes and balls

Roman Karasev [1] (joint with Arseniy Akopyan and Alfredo Hubard)

[1]Moscow Institute of Physics and Technology

Maykop

Theorem (Borsuk–Ulam theorem for continuous maps, 1933)

Any continuous map $f : \mathbb{S}^n \to \mathbb{R}^n$ maps some pair of opposite points x and $-x$ to the same point.

Theorem (Borsuk–Ulam theorem for odd maps, 1933)

Any odd continuous map $f : \mathbb{S}^n \to \mathbb{R}^n$ maps some pair of opposite points x and $-x$ to zero.

Odd maps are maps satisfying $f(-x) = -f(x)$.

Corollary

Assume that $f : \mathbb{S}^n \to \mathbb{R}^m$ *is an odd continuous map and* $Z = f^{-1}(0)$ *is a smooth* $(n-m)$*-dimensional manifold. Then* $\operatorname{vol}_{n-m} Z \ge \operatorname{vol}_{n-m} \mathbb{S}^{n-m}$.

The previous result uses the Crofton formula for the Riemannian volume of m-dimensional submanifolds $M \subset \mathbb{S}^n$,

$$\operatorname{vol}_m M = c_{n,m} \int_S \#(M \cap S) dS,$$

where the integral is taken over all $(n-m)$-dimensional subspheres $S \subset \mathbb{S}^n$ of unit radius using a rotation-invariant measure on the (Grassmannian) space of such subspheres, $c_{n,m} = \operatorname{vol}_m \mathbb{S}^m/2$ is a normalization coefficient.

Now we consider a density ρ on $\mathbb{R}^n$ and for k-dimensional submanifolds $M \subset \mathbb{R}^n$ define the *weighted Riemannian volume* through the ordinary Riemannian volume:

$$\operatorname{vol}_k^\rho M = \int_M \rho \sqrt{\det g}.$$

A particular case of interest is when ρ is a Gaussian density, which we prefer to have in the form $e^{-\pi|x|^2}$.

Corollary

Let ρ be a radially symmetric density in $\mathbb{R}^n$. Assume that $f : \mathbb{R}^n \to \mathbb{R}^m$ is an odd continuous map and $Z = f^{-1}(0)$ is a smooth $(n-m)$-dimensional manifold. Then

$$\mathrm{vol}^{\rho}_{n-m} Z \geq \mathrm{vol}^{\rho}_{n-m} \mathbb{R}^{n-m}.$$

Corollary (A particular case of the previous one)

Assume that $f : B^n \to \mathbb{R}^m$ *is an odd continuous map from the unit ball and* $Z = f^{-1}(0)$ *is a smooth* $(n-m)$*-dimensional manifold. Then*

$$\mathrm{vol}_{n-m} Z \geq \mathrm{vol}_{n-m} B^{n-m}.$$

Let us make a transportation of the Gaussian measure with density $\rho = e^{-\pi|x|^2}$. The map $T : \mathbb{R}^n \to (-1/2, 1/2)^n$, given by the coordinates

$$T_i(x) = \int_0^{x_i} e^{-\pi t^2}\, dt$$

evidently transports the Gaussian measure to the uniform measure in the cube and is 1-Lipschitz, $|T(x) - T(y)| \le |x - y|$.

A simple linear algebra exercise shows that when the derivative DT is restricted to a k-dimensional linear subspace $L \subseteq \mathbb{R}^n$ at a point $p \in \mathbb{R}^n$, then $\det DT|_L \ge \det DT = \rho(p)$. This means

$$\operatorname{vol}_k^\rho M \le \operatorname{vol}_k T(M).$$

Corollary (Akopyan, Hubard, K., 2017)

Assume that $f : (-1/2, 1/2)^n \to \mathbb{R}^m$ *is an odd continuous map and* $Z = f^{-1}(0)$ *is a smooth* $(n-m)$*-dimensional manifold. Then*

$$\operatorname{vol}_{n-m} Z \geq 1.$$

The case of a linear map is known as Vaaler's theorem (1979) on the sections of the cube. Vaaler actually used the "more peaked" technique different from the 1-Lipschitz transportation technique presented here.

If the considered maps are not odd, we need to leave a possibility to choose a fiber of the map $f^{-1}(y)$:

Theorem (Gromov's waist of the sphere theorem, 2003)

Any continuous map $f : \mathbb{S}^n \to \mathbb{R}^m$ *has a fiber* $f^{-1}(y)$ *with lower Minkowski content* $\underline{\mathcal{M}}_{n-m} f^{-1}(y) \geq \underline{\mathcal{M}}_{n-m} \mathbb{S}^{n-m}$.

Theorem (Gromov's waist theorem for the Gaussian measure, 2003)

Any continuous map $f : \mathbb{R}^n \to \mathbb{R}^m$ has a fiber $f^{-1}(y)$ with lower Minkowski content with respect to the radially symmetric Gaussian density ρ

$$\underline{\mathcal{M}}^{\rho}_{n-m} f^{-1}(y) \geq \underline{\mathcal{M}}^{\rho}_{n-m} \mathbb{R}^{n-m}.$$

Definition

For $X \subset \mathbb{R}^n$,

$$\underline{\mathcal{M}}_k^\rho X = \liminf_{t \to +0} \frac{\int_{\nu_t X} \rho \, dx}{v_{n-k} t^{n-k}},$$

where $\nu_t X$ is the metric t-neighborhood of X and v_{n-k} is the volume of the $(n-k)$-dimensional Euclidean ball.

For smooth m-dimensional submanifolds $X \subset \mathbb{R}^n$, this coincides with the ρ-weighted Riemannian volume. For the sphere, we consider the volume of the neighborhood in its intrinsic metric in the definition.

Remark. Gromov did not only establish the result for the limit, but established a tight estimate for any t-neighborhood for $t > 0$, namely

$$\mathrm{vol}^{\rho}\left(\nu_t f^{-1}(y)\right) \geq \mathrm{vol}^{\rho}\left(\nu_t \mathbb{R}^{n-m}\right)$$

in the Gaussian version.

The subsequent results will not work in this generality and in fact we have counterexamples to possible generalizations.

The already mentioned 1-Lipschitz transportation map $T : \mathbb{R}^n \to (-1/2, 1/2)^n$ has the property (following directly from definition)

$$\underline{\mathcal{M}}_k^\rho X \le \underline{\mathcal{M}}_k\, T(X).$$

In composition with Gromov's waist of the Gaussian measure theorem this implies:

Theorem (Klartag, 2016)

Any continuous map $f : (0,1)^n \to \mathbb{R}^m$ has a fiber $f^{-1}(y)$ with lower Minkowski content $\underline{\mathcal{M}}_{n-m}\, f^{-1}(y) \ge 1$.

Considering the natural projection $P : \mathbb{S}^{n+1} \to B^n$ instead of T, which is also 1-Lipschitz and pushes forward the uniform measure on the sphere to the uniform measure on the ball (as was noted by Archimedes), we obtain:

Theorem (Akopyan and K., 2016)

Any continuous map $f : B^n \to \mathbb{R}^m$ has a fiber $f^{-1}(y)$ with lower Minkowski content $\underline{\mathcal{M}}_{n-m} f^{-1}(y) \geq v_{n-m}$.

It is possible to understand the behavior of the Minkowski content under affine maps of $\mathbb{R}^n$. By the polar decomposition problem reduces to a diagonal scaling with factors $0 < a_1 \leq a_2 \leq \cdots \leq a_n$. The fact is that the lower (or upper) k-dimensional Minkowski content increases at least $a_1 a_2 \ldots a_k$ times.

Unfortunately, we have no (and could not find in the literature) full understanding of the behavior of the weighted Minkowski content under smooth maps.

Corollary (Klartag, Akopyan, K., 2016)

Consider a parallelotope $P = (0, a_1) \times \dots \times (0, a_n)$ *with dimensions* $0 < a_1 \le a_2 \le \dots \le a_n$ *and a continuous map* $f : P \to \mathbb{R}^m$. *There exists a fiber* $f^{-1}(y)$ *with* $\underline{\mathcal{M}}_{n-m} f^{-1}(y) \ge a_1 a_2 \dots a_{n-m}$.

Corollary (Akopyan and K., 2016)

Consider an ellipsoid E with principal axes $a_1 \le a_2 \le \dots \le a_n$ and a continuous map $f : E \to \mathbb{R}^m$. There exists a fiber $f^{-1}(y)$ with $\underline{\mathcal{M}}_{n-m} f^{-1}(y) \ge v_{n-m} a_1 a_2 \dots a_{n-m}$.

Corollary (Akopyan, Hubard, K., 2016)

Let $T_{a_1,\dots,a_n}$ be the orthogonal torus with dimensions $a_1 \le \dots \le a_n$. For any continuous map $f : T_{a_1,\dots,a_n} \to N$, where N is an m-dimensional manifold, there exists a fiber $f^{-1}(y)$ with

$$\underline{\mathcal{M}}_{n-m}\, f^{-1}(y) \ge a_1 a_2 \dots a_{n-m}.$$

A precise bound in this theorem is obtained because we allow arbitrary manifold in the codomain; the bound is attained at the natural projection $T_{a_1,\dots,a_n} \to T_{a_{n-k+1},\dots,a_n}$. In the classical formulation with $\mathbb{R}^m$ in place of arbitrary manifold N we only have the upper bound $2a_1a_2\dots a_{n-k}$ and do not know the truth.

Studying the results of this kind we naturally come to the question: Is it true that the Minkowski content in $\mathbb{R}^n$ decreases under 1-Lipschitz maps?

This is in fact a particular case (equal balls) of the famous Kneser–Poulsen conjecture: Is it true that the volume of the union of several equal balls decreases when we replace them with another system of corresponding balls so that all pairwise distances between the centers of the balls become smaller?

The Kneser–Poulsen conjecture is only established for $\mathbb{R}^2$ (Bezdek–Connelly), and for the case when the centers of the balls move continuously so that all distances between their centers decrease (Csikós, Gromov). The general case is open.

But we provide an alternative definition of the Gaussian Minkowski content that has the 1-Lipschitz and other good properties: For a compact set $X \subseteq \mathbb{R}^n$, define the lower *Gaussian Minkowski content*

$$\underline{\mathcal{G}}_k(X) = \liminf_{t \to +0} t^{n-k} \int_{\mathbb{R}^n} e^{-\pi t^2 \operatorname{dist}(x,X)^2} dx.$$

It is possible to check that for smooth k-dimensional submanifolds this coincides with the k-dimensional Riemannian volume.

The results of K. and Akopyan (2017) show that the waist of the balls in the model constant curvature spaces is also attained on their totally geodesic sections that are balls of smaller dimension.

The proofs are base on transformations of the weighted Riemannian manifolds and careful use of Gromov's pancake argument from the original proof of waist theorems.

Another result holds for arbitrary convex body, considered as a unit ball of a norm.

Theorem (Akopyan and K., 2017)

Suppose $K \subset \mathbb{R}^n$ *is a convex body and* $f : K \to \mathbb{R}^m$ *is a continuous map. Then for any* $t \in [0, 1]$ *there exists* $y_t \in \mathbb{R}^m$ *such that*

$$\operatorname{vol}(f^{-1}(y_t) + tK) \geq t^{n-m} \operatorname{vol} K.$$

Note the exchange of quantifiers $\forall$ and $\exists$ here; the corresponding Gromov-like statement fails if we do not exchange the quantifiers!

Theorem (Akopyan, Hubard, K., 2016)

For any odd continuous map $f : \mathbb{S}^n \to \mathbb{R}^k$, the Hausdorff measure

$$\mathcal{H}^{n-k}(f^{-1}(0)) \geq \mathcal{H}^{n-k}(\mathbb{S}^{n-k}).$$

Theorem (Akopyan, Hubard, K., 2016)

There exists a constant $\varepsilon_{m,n} > 0$ *such that for every continuous map*

$$f : [0,1]^n \to Y$$

from the unit cube $[0,1]^n$ *to an m-dimensional polyhedron* Y *there exists* $y \in Y$ *such that the set* $f^{-1}(y)$ *has* $(n-m)$*-Hausdorff measure* $\mathcal{H}^{n-m}(f^{-1}(y))$ *at least* $\varepsilon_{m,n}$*.*

More theorems on Gromov-type waist can be found in:

arXiv:1608.04121

arXiv:1608.06279

arXiv:1612.06926

arXiv:1702.07513