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The Lipschitz property of the metric projection in the Hilbert space

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Quasilinear Equations, Inverse Problems and Their Applications

Let $\mathcal{H}$ be a real Hilbert space. For a set $A \subset \mathcal{H}$ and a point $x \in \mathcal{H}$ put
$P_A x = \{a \in A \mid \|x - a\| = \inf_{y \in A} \|x - y\|\}$, $U_A(r) = \{x \in \mathcal{H} \mid \inf_{y \in A} \|x - y\| < r\}$,
$AP_A x = \{a \in A \mid \|x - a\| = \sup_{y \in A} \|x - y\|\}$, $T_A(r) = \{x \in \mathcal{H} \mid \sup_{y \in A} \|x - y\| > r\}$.

The Lipschitz solution $x \to P_A x$ of the problem

$$\inf_{a \in A} \|x - a\|, \tag{1}$$

The Lipschitz solution $x \to AP_A x$ of the problem

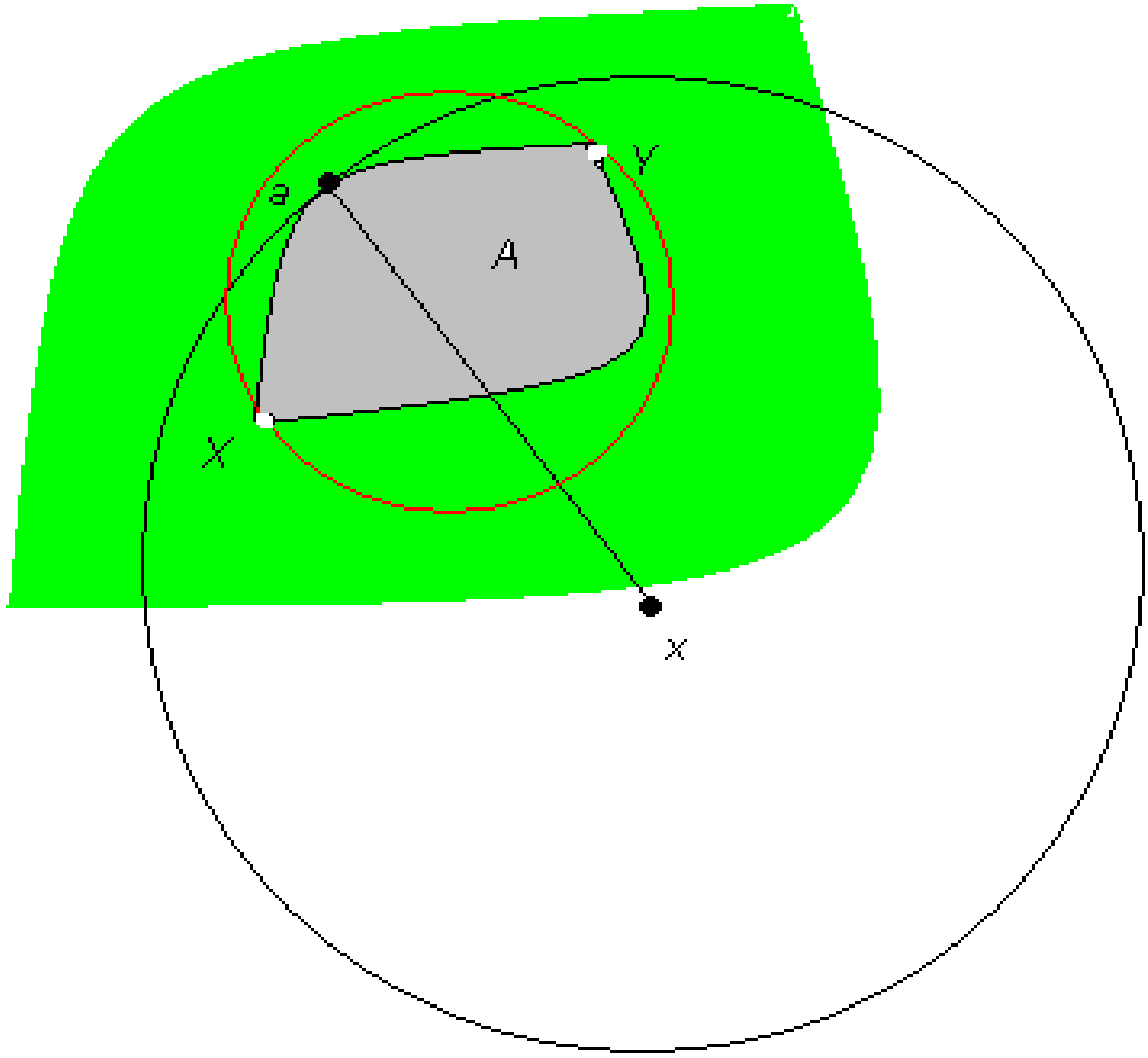
$$\sup_{a \in A} \|x - a\|. \tag{2}$$

Theorem 1. *Let* $A \subset \mathcal{H}$ *be a closed convex subset. Then the metric projector operator is Lipschitz continuous with constant* $C \in (0,1)$ *on the set* $\mathcal{H} \backslash U_A(r)$ *(i.e. for all* $x_1, x_2 \in \mathcal{H} \backslash U_A(r)$ *we have*

$$\|P_A x_1 - P_A x_2\| \leq C \|x_1 - x_2\|)$$

if and only if the set A *is intersection of closed balls of radius* $R = \frac{Cr}{1-C}$.

Applications: Characterization of the Hilbert space, gradient projection algorithm.

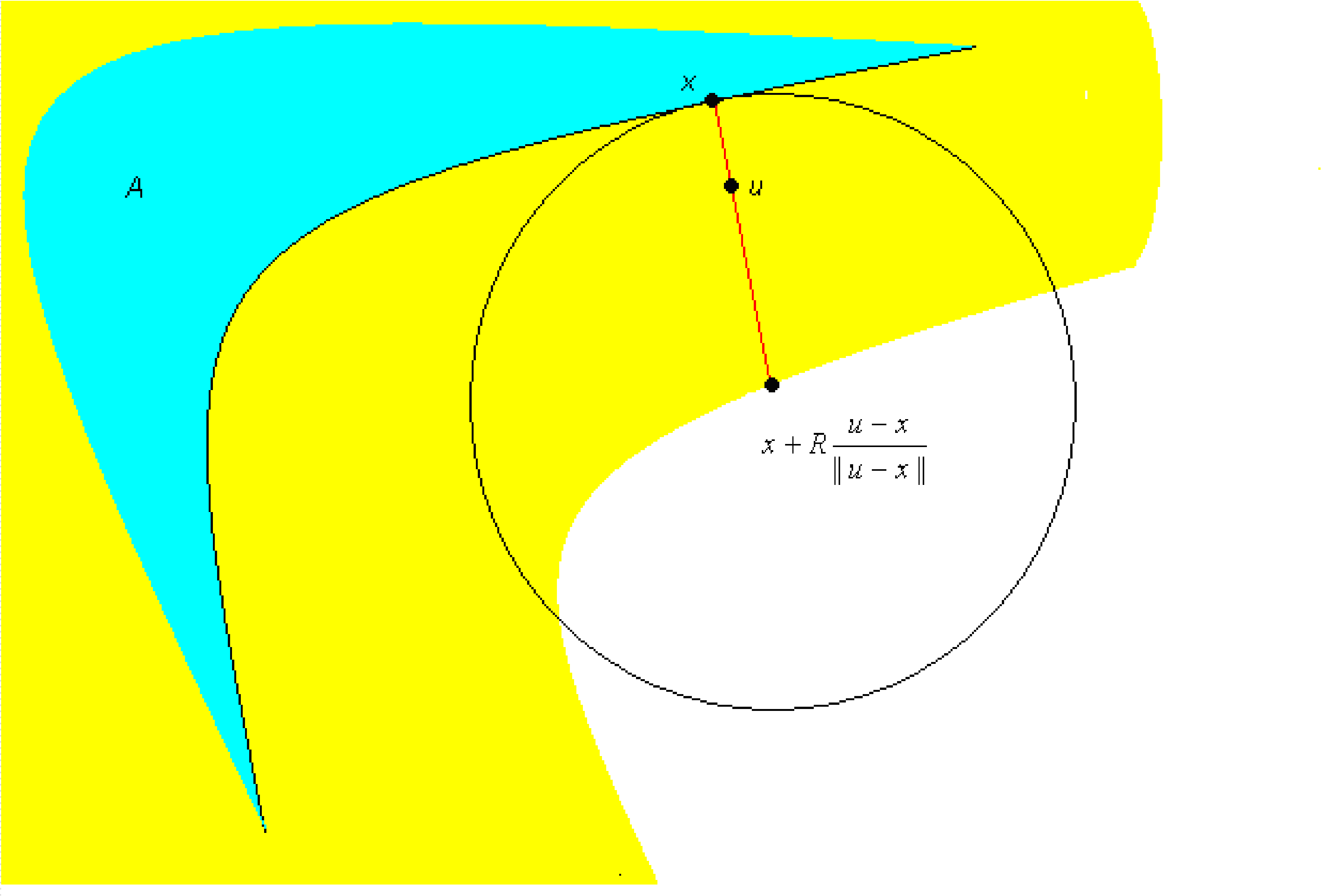


a
Y
A
X
x

Theorem 2. *Let $r > 0$, $A \subset \mathcal{H}$ be a closed bounded convex subset. The set A is intersection of closed balls of radius R if and only if for any points $x_1, x_2 \in T_A(r)$*

$$\|AP_A x_1 - AP_A x_2\| \leq \frac{R}{r-R}\|x_1 - x_2\|.$$

A closed subset $A \subset \mathcal{H}$ is called proximally smooth (=prox regular) with constant $R > 0$ if the distance function $x \to \varrho_A(x) = \inf_{a \in A} \|x - a\|$ is (Frechet) differentiable on the set $U_A(R)$ (Rockafellar, Clarke, Borwein, Stern, Wolenski, Ledyaev...).



x
A
u
$x + R\frac{u-x}{\|u-x\|}$

Theorem 3. *Let $r > 0$, $A \subset \mathcal{H}$ a closed subset with the property*

$$\exists C > 1 \ \forall x_1, x_2 \in U_A(r) \quad \|P_A x_1 - P_A x_2\| \leq C\|x_1 - x_2\|.$$

Then the set A is proximally smooth with constant $R = \frac{C}{C-1}r$.

Together with the result

Proposition 1 (Clarke, Stern, Wolenski). *Let a closed subset $A \subset \mathcal{H}$ is proximally smooth (=prox regular) with constant $R > 0$, $0 < r < R$. Then $\forall x_1, x_2 \in U_A(r)$*

$$\|P_A x_1 - P_A x_2\| \leq \frac{R}{R - r}\|x_1 - x_2\|.$$

Theorem 3 gives the best possible relationship between parameters r, R, C.

Application. Let $r > 0$, E be a uniformly convex and uniformly smooth Banach space and a closed subset A has the Lipschitz continuous metric projection $P_A x$ for all $x \in U_A(r)$ with the Lipschitz constant $C = 1$. Then the set A is convex.

Thank you!