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Polyhedral approximations in R^n

22-27 May 2017, Saint-Petersburg

Constructive Nonsmooth Analysis and Related Topics

Dedicated to the memory of Professor V.F. Demyanov

Approximation

Let $\mathbb{R}^n$ be a real n-dimensional Euclidean space with the inner product (x, y).

We shall consider the standard polyhedral approximation of a closed convex compactum $A \subset \mathbb{R}^n$ on a grid $\mathbb{G} = \{p_k\}_{k=1}^N$ of unit vectors from $\mathbb{R}^n$ with step $\Delta \in (0, \frac{1}{2})$:

$$\hat{A} = \{x \in \mathbb{R}^n \mid (p_k, x) \leq s(p_k, A), \ \forall k, \ 1 \leq k \leq N\}.$$

Here $s(p, A)$ is the supporting function of the set A, i.e.

$$s(p, A) = \sup_{x \in A} (p, x).$$

Grid (E. Polovinkin)

A *grid* $\mathbb{G}$ with *step* $\Delta \in (0, \frac{1}{2})$ is a finite collection of unit vectors $\{p_i\} \subset \mathbb{R}^n$, $i \in \overline{1, I} = \{1, \ldots, I\}$, such that for any vector $p \neq 0$, $p \in \mathbb{R}^n$, with $\frac{p}{||p||} \notin \mathbb{G}$ there exist a set of indexes $I_p \subset \overline{1, I}$ and numbers $\alpha_i > 0$, $i \in I_p$, with the property

$$p = \sum_{i \in I_p} \alpha_i p_i, \qquad p_i \in \mathbb{G},$$

$$\|p_i - p_j\| < \Delta, \qquad \forall i, j \in I_p.$$

Moreover, for

$$\hat{p} = \frac{p}{\alpha} = \sum_{i \in I_p} \hat{\alpha}_i p_i, \qquad \alpha = \sum_{i \in I_p} \alpha_i, \quad \hat{\alpha}_i = \frac{\alpha_i}{\alpha}$$

$$\|\hat{p} - p_j\| < \Delta, \quad \forall j \in I_p, \qquad 1 \geq \|\hat{p}\| \geq 1 - \frac{1}{2}\Delta^2,$$

Known results

1) for an arbitrary convex compactum $h(A, \hat{A}) \le C \operatorname{diam} A \cdot \Delta$,

2) for a set $A = \bigcap_{x \in X} B_R(x)$ $h(A, \hat{A}) \le CR \cdot \Delta^2$ (E. Polovinkin),

3) for a set A with modulus of convexity $\delta_A(\cdot)$ $h(A, \hat{A}) \le \frac{8}{7}\varepsilon(\Delta)\Delta$,
where $\varepsilon(\Delta)$ is a solution of the equation $\frac{\delta_A(\varepsilon)}{\varepsilon} = \frac{\Delta}{4-\Delta^2}$.

4) if $\hat{A} = \{x \mid (p_i, x) \le f(p) \ \forall p_i \in \mathbb{G}\}$ and f is nonconvex, then the estimate for convex case (when $f(p) = s(p, A)$) should be multiplied on R/r (sometimes on R^2/r), where $R > r > 0$ such numbers that $r\|p\| \le f(p) \le f(p)$ $\forall \|p\| = 1$.

Assumptions

For a positively uniform function $f : \mathbb{R}^n \to \mathbb{R}$ define the set

$$O_f = \{p \in \mathbb{R}^n \mid ||p|| = 1,\ \mathrm{conv} f(p) = f(p)\}.$$

We shall postulate that $\mathrm{conv} f(p)$ is a proper function (and $f(p)$ itself is not necessary convex).

Let $R \geq r > 0$ be such constants that $r||p|| \leq f(p) \leq R||p||$ for all p. Suppose also that there exists $\delta > 0$ such that for any $p \in O_f$ and for all $q \in B_\delta(p) \cap \partial B_1(0)$ we have

$$f(q) - f(p) - (f'(p), q - p) \leq w(||q - p||),$$

where $f'(p)$ is some subgradient, $w(t) > 0$ for $t \in (0, \delta)$ and $\lim\limits_{t \to +0} \frac{w(t)}{t} = 0$.

Result

Let

$$A = \{x \in \mathbb{R}^n \mid (p, x) \leq f(p), \ \forall p \in \mathbb{R}^n\},$$

$$\tilde{A} = \{x \in \mathbb{R}^n \mid (p, x) \leq f(p), \ \forall p \in \mathbb{G}\}.$$

Then for sufficiently small step Δ we have the estimate

$$h(A, \tilde{A}) \leq \frac{R}{r} \frac{w(\Delta)}{1 - \frac{\Delta^2}{2}}.$$

The order (on Δ) of the right side in the previous formula is the best possible.

Thank you!