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A New Version of the Discrete Ordinate Method for the Calculation of the Intrinsic Radiation in Horizontally Homogeneous Atmospheres

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Abstract—A new version of the discrete ordinate method for the calculation of the transfer of monochromatic radiation in a scattering, absorbing, and emitting plane-parallel atmosphere of Earth and other planets is proposed. A feature of this version is that the system of linear equations obtained by the discrete ordinate method is solved using the block elimination method. This is an exact and computationally efficient method; moreover it is easy to implement. The computer program developed based on this method is about two times faster than the program used in the free DISORT package.

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1. INTRODUCTION

In many applications of physics, there is need in calculating the intrinsic radiation of the Earth atmosphere and the atmospheres of other planets and their satellites under the assumption that the atmosphere is horizontally homogeneous and its intrinsic radiation depends only on the altitude and the zenith angle. For instance, this is needed when the rate of atmosphere heating and cooling due to the radiation transfer is calculated or when data obtained by remotely measured atmospheric parameters are interpreted. The field of the monochromatic radiation in a scattering, absorbing, and emitting plane-parallel atmosphere is described by the one-dimensional radiation transfer equation subject to appropriate boundary conditions. This equation is numerically solved using various versions of the Monte Carlo method, discrete ordinate method (DOM), the successive order of scattering method, and the spherical harmonic method (see [1–4]).

The Monte Carlo method was successfully used in the problem under consideration in [5, 6], and the foundations of this method are presented, e.g., in [1, 2, 4]. This is an approximate method, and it is convenient for parallel computations. A feature of this method is that its accuracy is proportional to $1/\sqrt{N}$, where N is the number of computed photon trajectories. In addition, in the presence of optically thick layers in which the single-scattering albedo is close to unity (e.g., the cloud layer of the Venus atmosphere), the number of computed photon trajectories can be considerably greater than in the case when there are no optically thick layers. The successive order of scattering method is also approximate. It is efficient only when there are no optically thick atmospheric layers (the optical thickness is greater than 0.5) in which the single-scattering albedo is greater than 0.6–0.7. Otherwise, the iterations in this method converge very slowly.

The DOM for the calculation of the transfer of monochromatic radiation in a scattering, absorbing, and emitting plane-parallel atmosphere is as follows. A grid with respect to the zenith angle is introduced. The specific intensity of radiation is subdivided into a finite number of fluxes of which each is associated with a fixed zenith angle of the grid. The integral over the angles that specifies the scattered radiation source is approximated by a linear combination of fluxes. The radiation transfer equation is replaced with a system of ordinary differential equations that describe the variation of radiation with the given zenith angle depending on the altitude. Next, sampling with respect to the altitude is performed. The atmosphere is represented by M horizontal layers in which the scattering indicatrix and the single-scattering albedo are assumed to be either constant (but they can vary from layer to layer) or depending on the optical thickness (the dependence may vary from layer to layer). The system of differential equations is replaced by a system of integral equations with respect to the altitude that relate the radiation intensities at the zenith angle grid nodes in the adjacent horizontal layers. Then, the system of integral equations is replaced by a system of linear algebraic equations in the radiation intensities at the zenith angle and altitude grid nodes. This replacement is done by approximating the integral over the altitude in the integral equations using analytical formulas.

Various versions of the DOM differ in the approximation of integrals over the altitude and scattering angles and in the method used to solve simultaneous linear algebraic equations. In the free software package DISORT (see [7]), the integrals over scattering angles are approximated using the Gauss-Legendre quadrature formulas and the system of linear algebraic equations is solved by finding the eigenvalues and eigenvectors of the coefficient matrix.

There is also the free package SHDOM (see [8]). It uses a combined spherical harmonics and discrete ordinate method. This procedure uses an iterative technique for calculating the function of the scattered radiation source at the nodes of the spatial grid, and the angular part of the source function is represented by a finite sum of spherical harmonics. This process is equivalent to the successive order of scattering method. The number of iterations needed to achieve the required accuracy increases with increasing single-scattering albedo and optical thickness.

In this paper, we propose a new modification of DOM for the calculation of the transfer of monochromatic radiation in a horizontally homogeneous atmosphere. This modification has two specific features. The first one is that the computational grid with respect to zenith angles may be arbitrary. The second feature is that the system of linear algebraic equations that arises in the DOM is solved using the block elimination method. This is an exact method that makes extensive use of the structure of the coefficient matrix to reduce the amount of computations. It is more efficient and simpler for implementation than the method used in DISORT, which is based on the computation of eigenvalues and eigenvectors of a large coefficient matrix. If the atmosphere includes layers with strong scattering and weak absorption (e.g., cloud layers in the atmospheres of Earth and Venus), iterative methods can converge slowly and require a large number of iteration steps to ensure the acceptable accuracy of the solution. In this case, the method proposed in the present paper has an advantage in terms of accuracy and computation speed.

2. STATEMENT OF THE PROBLEM FOR THE CALCULATION OF INTRINSIC ATMOSPHERIC RADIATION

We assume that the atmosphere is flat and horizontally homogeneous. Consider the monochromatic radiation in a scattering, absorbing, and emitting plane-parallel atmosphere at the frequency ν. We assume that this radiation depends only on the altitude above the surface and on the direction of the photon momentum and the vertical direction. Sometimes this angle is counted from the downward direction.

Let us introduce the following notation: u is the cosine of the zenith angle, z is the altitude above the ground; $z_{\max}$ is the altitude of the upper boundary of the atmospheric column for which the radiation field is calculated; $T(z)$ is the temperature of the atmospheric gas; $I(z, u)$ is the intensity of monochromatic radiation at the frequency ν, the zenith angle with the cosine u, at the altitude z; $B(T, \nu) = 2h\nu^3c^{-2}(\exp(h\nu/(k_BT)) - 1)^{-1}$ is the Planck function in which h is the Planck constant, k_B is the Boltzmann constant, and c is the speed of light.

Under these assumptions, the transport equation for the monochromatic radiation can be written as (see, e.g., [1–3])

$$\frac{u}{\sigma(z)}\frac{dI(z,u)}{dz} = -I(z,u) + (1-\omega(z))B((T(z)),\nu) + S(z,u), \tag{1}$$

where $\sigma(z)$ and $\omega(z)$ are the strictly positive extinction coefficient and single-scattering albedo at the altitude z for the monochromatic radiation at the frequency ν, respectively; $S(z, u)$ is the renormalized density of the source of scattered radiation at the frequency ν at the altitude z and the zenith angle with the cosine u. This density is given by the formula

$$S(z,u) = \frac{\omega(z)}{4\pi}\int_{-1}^{1} I(z,\omega)\left(\int_0^{2\pi}\chi(z, v(w,u,\varphi))d\varphi\right)dw, \tag{2}$$

where w and u are the cosines of the zenith angles before and after scattering, φ is the difference between the azimuthal angle of radiation before scattering and the same angle after scattering, $v(w, u, \varphi) =$

$uw + \cos\varphi\sqrt{(1-u^2)(1-w^2)}$ is the cosine of scattering angle, $\chi(z, v)$ is the scattering indicatrix for the radiation at the frequency ν at the altitude z by the angle with the cosine v.

The optical thickness of the atmospheric layer between the upper boundary and the altitude z is defined as the parameter

$$\tau(z) = \int_z^{z_{\max}} \sigma(z)dz, \quad d\tau = -\sigma(z)dz, \quad \tau(z_{\max}) = 0, \quad \tau(0) = \tau_{\max}.$$

which is in one-to-one correspondence with z. By replacing the dependence on z with the dependence on τ in all the functions of Eq. (1), we can rewrite this equation as

$$u\frac{dI(\tau, u)}{d\tau} = I(\tau, u) - (1-\omega(\tau))B(T(\tau), \nu) - S(\tau, u). \tag{3}$$

Equation (3) should be complemented with boundary conditions. The standard condition on the upper boundary for the intrinsic radiation of the atmosphere is that the intensity of downward radiation is zero:

$$I(u) = 0, \quad u < 0, \quad \tau = 0. \tag{4}$$

On the lower boundary, the intensity of upward radiation is composed from the intensity of downward radiation scattered by the surface and the intensity of thermal emission of the surface given by $(1 - \Omega(\nu))B(T_p, \nu)$, where T_p is the surface temperature and $\Omega(\nu)$ is the albedo of the surface for the radiation at the frequency ν. If the scattering by the surface is isotropic, then the condition on the lower boundary can be written as

$$I(u) = (1-\Omega(\nu))B(T_p, \nu) + \Omega(\nu)\int_{-1}^{0} I(w)dw, \quad u \ge 0, \quad \tau = \tau_{\max}. \tag{5}$$

It is seen from (5) that the intensity of the upward radiation on the lower boundary is independent of u.

Expand the scattering indicatrix in Legendre polynomials

$$\chi(\tau, u) = \sum_{k=0}^{\infty} a_k(\tau)P_k(u), \tag{6}$$

where $P_k(u)$ is the kth Legendre polynomial, substitute (6) into the right-hand side of (2), and use the equality

$$\int_0^{2\pi} P_k(uw + \cos\varphi\sqrt{(1-u^2)(1-w^2)})d\varphi = 2\pi P_k(u)P_k(w),$$

which is a consequence of the summation theorem for Legendre polynomials (see [1–3]), to obtain the following formula for the renormalized density of the scattered radiation source

$$S(\tau, u) = \frac{\omega(\tau)}{2}\sum_{k=0}^{\infty} a_k(\tau)P_k(u)\int_{-1}^{1} I(\tau, w)P_k(w)dw. \tag{7}$$

3. DISCRETIZATION WITH RESPECT TO THE ZENITH ANGLE AND DERIVATION OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Define an arbitrary grid with respect to the cosines of the zenith angles such that it contains an even number of nodes enumerated in ascending order and satisfies the conditions $-1 = u_0 < u_1 < u_2 < \ldots < u_N = 1$, $u_i < 0$ for $i \le (N-1)/2$, and $u_i > 0$ for $i \ge (N+1)/2$. For the given odd N, the highest accuracy is achieved when the zenith angle grid is uniform and given by the formula $u_i = \cos(\pi(i/N - 1))$. If the Gauss–Legendre quadrature formulas are used to approximate the integrals over the scattering angles, the nodes u_i of this grid are roots of the Legendre polynomial with the index $N + 1$. Such a grid is used in DISORT [7].

We approximate the radiation field by a linear combination of the Legendre polynomials with the indices from 0 through N:

$$I(\tau, u) \approx \sum_{k=0}^{N} b_k(\tau) P_k(u), \tag{8}$$

We also require that this combination coincide with the values of the radiation intensity $I(\tau, u)$ at the grid nodes u_i. Then, the coefficients $b_k(\tau)$ are determined by the system of linear algebraic equations

$$I(\tau, u_i) = \sum_{k=0}^{N} b_k(\tau) P_k(u_i), \quad i = 0, 1, \ldots, N. \tag{9}$$

If we substitute (8) into (7) and use the pairwise orthogonality and normalization of the Legendre polynomials, then we obtain an approximation for the renormalized density of the scattered radiation source at the grid points u_i:

$$S(\tau, u_i) \approx \omega(\tau) \sum_{k=0}^{N} \frac{1}{2k+1} a_k(\tau) b_k(\tau) P_k(u_i). \tag{10}$$

At each altitude z, we introduce the column vector of expansion coefficients (8)

$$\mathbf{b}(\tau) = (b_0(\tau), \ldots, b_N(\tau))^{\mathrm{T}}, \quad \tau = \tau(z),$$

and column vectors of radiation intensity and density of the scattered radiation source at the grid points u_i:

$$\mathbf{I}(\tau) = (I(\tau, u_0), \ldots, I(\tau, u_N))^{\mathrm{T}}, \quad \mathbf{S}(\tau) = (S(\tau, u_0), \ldots, S(\tau, u_N))^{\mathrm{T}}.$$

The system of linear algebraic equations (9) can be written in vector form $\mathbf{I}(\tau) = \hat{\mathbf{L}}\mathbf{b}(\tau)$, where $\hat{\mathbf{L}}$ is the matrix of size $(N + 1) \times (N + 1)$ whose elements L_{jk} are given by the formula $L_{jk} = P_k(u_j)$ for $j, k = 0, \ldots, N$. Note that the matrix $\hat{\mathbf{L}}$ is invertible because its columns are linearly independent. Denote by $\hat{\mathbf{L}}^{-1}$ the inverse of $\hat{\mathbf{L}}$. The elements of $\hat{\mathbf{L}}^{-1}$ are denoted by L_{ji}^{inv}. Using this matrix, the solution to (9) can be written in vector and scalar form as

$$\mathbf{b}(\tau) = \hat{\mathbf{L}}^{-1}\mathbf{I}(\tau), \quad b_k(\tau) = \sum_{i=0}^{N} L_{ki}^{\mathrm{inv}} I(\tau, u_i). \tag{11}$$

Substitute (11) into (10) to obtain

$$S(\tau, u_j) \approx \omega(\tau) \sum_{k=0}^{N} L_{jk} \frac{a_k(\tau)}{2k+1} \sum_{i=0}^{N} L_{ki}^{\mathrm{inv}} I(\tau, u_i). \tag{12}$$

Define the matrix $\hat{\mathbf{R}}(\tau)$ whose elements $R_{ji}(\tau)$ are given by

$$R_{ji}(\tau) = \omega(\tau) \sum_{k=0}^{N} L_{jk} \frac{a_k(\tau)}{2k+1} L_{ki}^{\mathrm{inv}}, \quad j, i = 0, \ldots, N. \tag{13}$$

$R_{ji}(\tau)$ is equal to the ratio of the intensity of radiation whose cosine of the zenith angle at the scattering altitude $z(\tau)$ changed from u_i to u_j to the intensity of the radiation absorbed and scattered at this altitude whose cosine of the zenith angle before absorption and scattering was equal to u_i.

Using the matrix $\hat{\mathbf{R}}(\tau)$, formula (12) can be written in vector form as

$$\mathbf{S}(\tau) \approx \hat{\mathbf{R}}(\tau)\mathbf{I}(\tau). \tag{14}$$

Equation (3) is replaced with the system of $N+1$ linear ordinary differential equations for the intensities $I(\tau, u_j)$:

$$u_j\frac{dI(\tau, u_j)}{d\tau} = I(\tau, u_j) - \sum_{i=0}^{N} R_{ji}(\tau)I(\tau, u_i) - (1-\omega(\tau))B(T(\tau), \nu). \tag{15}$$

By removing the term $R_{jj}(\tau)I(\tau, u_j)$ from the summation sign in (15), we can write these equations in the form

$$u_j\frac{dI(\tau, u_j)}{d\tau} = (1-R_{jj}(\tau))I(\tau, u_j) - \sum_{i=0}^{N}(1-\delta_{ji})R_{ji}(\tau)I(\tau, u_i) - (1-\omega(\tau))B(T(\tau), \nu), \tag{16}$$

where δ_{ji} is the Kronecker delta. The system of equations (16) should be complemented with conditions on the upper and lower boundaries. Condition (4) implies the condition for the downward radiation ($u_j < 0$) on the upper boundary:

$$I(u_j) = 0, \quad j = 0, \dots, (N-1)/2, \quad \tau = 0. \tag{17}$$

By approximating the integral

$$\int_{-1}^{0} I(\tau_{\max}, u)du \approx I(\tau_{\max}, u_0)\frac{u_1-u_0}{2} + \sum_{i=1}^{(N-1)/2} I(\tau_{\max}, u_i)\frac{u_{i+1}-u_{i-1}}{2}$$

and using condition (5), we obtain conditions for the intensity of upward radiation ($u_j > 0$) on the lower boundary:

$$I(\tau_{\max}, u_j) = (1-\Omega(\nu))B(T_p, \nu) + \Omega(\nu)\left(I(\tau_{\max}, u_0)\frac{u_1-u_0}{2} + \sum_{i=1}^{(N-1)/2} I(\tau_{\max}, u_i)\frac{u_{i+1}-u_{i-1}}{2}\right),$$
$$j = (N+1)/2, \dots, N. \tag{18}$$

4. DISCRETIZATION WITH RESPECT TO THE OPTIC THICKNESS AND DERIVATION OF A SYSTEM OF LINEARV ALGEBRAIC EQUATIONS

Consider the transition from the system of differential equations (16) to the system of integral equations relating the column vectors of radiation intensities $\mathbf{I}(\tau)$ in adjacent altitude layers. Define an arbitrary grid with respect to the altitude with the nodes enumerated in ascending order and with the total number of grid steps $M: 0 = z_0 < z_1 < \dots < z_M = z_{\max}$. The values of the optical thickness from the upper boundary at the grid points $\tau(z_k)$ form a grid with respect to the optical thickness on the interval $[0, \tau_{\max}]$. We enumerate its nodes in ascending order as $\tau_0 = \tau(z_M) = 0$, $\tau_k = \tau(z_{M-k})$, $\tau_M = \tau(z_0) = \tau_{\max}$.

At a fixed j, each equation in (16) is a linear ordinary first-order differential equation for $I(\tau, u_j)$

$$\frac{dI(\tau, u_j)}{d\tau} = \alpha(\tau)I(\tau, u_j) + \Phi(\tau)$$

with the coefficient $\alpha(\tau) = \frac{1}{u_j}(1-R_{jj}(\tau))$ and the right-hand side given by the formula

$$\Phi(\tau) = -\frac{1}{u_j}\sum_{i=0}^{N}(1-\delta_{ji})R_{ji}(\tau)I(\tau, u_i) - \frac{1}{u_j}(1-\omega(\tau))B(T(\tau), \nu).$$

The solution to the Cauchy problem for the last equation subject to the initial condition at the node τ_{k-1} is given by the Cauchy formula

$$I(\tau, u_j) = I(\tau_{k-1}, u_j)\exp\left(\int_{\tau_{k-1}}^{\tau}\alpha(s)ds\right) + \int_{\tau_{k-1}}^{\tau}\Phi(s)\exp\left(\int_{s}^{\tau}\alpha(y)dy\right)ds.$$

Substitute in the last formula τ_k for τ and plug expressions for $\alpha(\tau)$ and $\Phi(\tau)$ to obtain linear integral equations for $I(\tau_k, u_j)$ for $k = 1, ..., M$:

$$I(\tau_k, u_j) = I(\tau_{k-1}, u_j)\exp\left(\int_{\tau_{k-1}}^{\tau_k}(1-R_{jj}(\tau))\frac{1}{u_j}d\tau\right) - \int_{\tau_{k-1}}^{\tau_k} G_j(\tau)\exp\left(\int_{\tau}^{\tau_k}(1-R_{jj}(s))\frac{1}{u_j}ds\right)\frac{(1-R_{jj}(\tau))}{u_j}d\tau, \quad (19)$$

Here,

$$G_j(\tau) = \frac{1}{1-R_{jj}(\tau)}\left((1-\omega(\tau))B(T(\tau),\nu) + \sum_{i=0}^{N}(1-\delta_{ji})R_{ji}(\tau)I(\tau, u_i)\right). \quad (20)$$

An important feature of these equations is that the factor $(1 - R_{jj}(\tau))$ in the exponent analytically represents the fact that a part of the scattered radiation preserves the zenith angle.

Consider the transition from the system of integral equations (19) and boundary conditions (17), (18) to the system of linear algebraic equations for the radiation intensities at the zenith angle and optical thickness grid points. Define functions of the optical thickness

$$\mu_j(\tau) = \int_0^{\tau}(1-R_{jj}(\tau))\frac{1}{|u_j|}d\tau = \int_{z(\tau)}^{z_M}(1-R_{jj}(\tau(z)))\frac{\sigma(\tau(z))}{|u_j|}dz. \quad (21)$$

The functions $\mu_j(\tau)$ strictly increase with increasing τ, and there exist inverse functions $\tau(\mu_j)$. Below, we will use the notation

$$d\mu_j = \frac{\sigma(\tau)}{|u_j|}(1-R_{jj}(\tau))d\tau, \quad \mu_j(\tau_k) = \mu_{jk}, \quad (\Delta\mu_j)_{k-1/2} = \mu_{jk} - \mu_{jk-1}. \quad (22)$$

Using notation (20)–(22), Eqs. (19) for the intensities of downward radiation ($u_j < 0$) can be written as

$$I(\tau_k, u_j) = I(\tau_{k-1}, u_j)\exp(\mu_{jk-1} - \mu_{jk}) + \int_{\mu_{jk-1}}^{\mu_{jk}} G_j(\tau(\mu_j))\exp(\mu_j - \mu_{jk})d\mu_j, \quad (23)$$

for the intensities of upward radiation ($u_j > 0$), Eqs. (19) can be written as

$$I(\tau_{k-1}, u_j) = I(\tau_\kappa, u_j)\exp(\mu_{jk-1} - \mu_{jk}) + \int_{\mu_{jk-1}}^{\mu_{jk}} G_j(\tau(\mu_j))\exp(\mu_{jk-1} - \mu_j)d\mu_j. \quad (24)$$

In order to approximately calculate the integrals on the right-hand sides of Eqs. (23) and (24), we will approximate the integrand $G_j(\tau(\mu_j))$ by a linear function of $\mu_{jk-1} \le \mu_j \le \mu_{jk}$ as

$$G_j(\tau(\mu_j)) \approx G_j(\tau_{k-1}) + (G_j(\tau_k) - G_j(\tau_{k-1}))\frac{\mu_j - \mu_{jk-1}}{(\Delta\mu_j)_{k-1/2}}. \quad (25)$$

Substitute (25) into the integrals in Eqs. (23) and (24) and using the equalities $\int \mu e^{\pm\mu}d\mu = \pm e^{\pm\mu}(\mu - (\pm 1))$ + Const, we obtain for the integrals in (24) and (25) an approximate expressions that can be written as

$$\int_{\mu_{jk-1}}^{\mu_{jk}} G_j(\tau(\mu_j))\exp(\mu_j - \mu_{jk})d\mu_j \approx f_{jk-1/2}\, G_j(\tau_{k-1}) + W_{jk-1/2}\, G_j(\tau_k), \quad (26)$$

$$\int_{\mu_{jk-1}}^{\mu_{jk}} G_j(\tau(\mu_j))\exp(\mu_{jk-1} - \mu_j)d\mu_j \approx W_{jk-1/2}\, G_j(\tau_{k-1}) + f_{jk-1/2}\, G_j(\tau_k), \quad (27)$$

where the numbers $W_{jk-1/2}$ and $f_{jk-1/2}$ for $k = 1, ..., M$ are given by the formulas

$$W_{jk-1/2} = 1-\frac{1-V_{jk-1/2}}{(\Delta\mu_j)_{k-1/2}}, \quad V_{jk-1/2} = \exp(-(\Delta\mu_j)_{k-1/2}),$$
$$f_{jk-1/2} = 1-W_{jk-1/2}-V_{jk-1/2}. \tag{28}$$

According to (21) and (22), the numbers $(\Delta\mu_j)_{k-1/2}$ are determined by the formulas

$$(\Delta\mu_j)_{k-1/2} = \int\limits_{\tau_{k-1}}^{\tau_k}(1-R_{jj}(\tau))\frac{d\tau}{|u_j|} = \int\limits_{z_{M-k}}^{z_{M-k+1}}(1-R_{jj}(\tau(z)))\frac{\sigma(\tau(z))}{|u_j|}dz.$$

Various methods for the approximate calculation of these numbers may be used. If the single-scattering albedo and the indicatrix are constant within the layer $\tau_{k-1}<\tau<\tau_k$, then the elements of the matrix $\hat{\mathbf{R}}(\tau)$ are also constant within this layer, and $(\Delta\mu_j)_{k-1/2}$ can be approximately calculated by

$$(\Delta\mu_j)_{k-1/2} \approx \frac{\tau_k-\tau_{k-1}}{|u_j|}(1-R_{jj}(\tau_{k-1/2})). \tag{29}$$

If the elements of $\hat{\mathbf{R}}(\tau)$ are not constant within this layer, then the approximate expressions

$$(\Delta\mu_j)_{k-1/2} \approx \frac{\tau_k-\tau_{k-1}}{|u_j|}\left(1-\frac{\sigma(\tau_k)R_{jj}(\tau_k)+\sigma(\tau_{k-1})R_{jj}(\tau_{k-1})}{\sigma(\tau_{k-1})+\sigma(\tau_k)}\right)$$
$$\approx \frac{(z_{M-k+1}-z_{M-k})}{2|u_j|}(\sigma(\tau_k)(1-R_{jj}(\tau_k))+\sigma(\tau_{k-1})(1-R_{jj}(\tau_{k-1}))) \tag{30}$$

can be used.

Consider the case when the elements of $\hat{\mathbf{R}}(\tau)$ are constant within the layer $\tau_{k-1}<\tau<\tau_k$. Substitute formulas (26) and (20) into (23) to obtain for the intensities of downward radiation ($u_j<0$) the following linear algebraic equations for the intensities $I(\tau_k, u_j)$ at $k = 1, \ldots, M$:

$$I(\tau_k, u_j)-V_{jk-1/2}I(\tau_{k-1}, u_j)-W_{jk-1/2}\sum_{i=0}^{N}Y_{jik-1/2}I(\tau_k, u_i)-f_{jk-1/2}\sum_{i=0}^{N}Y_{ijk-1/2}I(\tau_{k-1}, u_i)$$
$$= H_{jk-1/2}(W_{jk-1/2}D_k+f_{jk-1/2}D_{k-1}), \tag{31}$$

In these equations, the notation

$$D_k = B(T(\tau_k), \nu), \quad k = 0, \ldots, M, \tag{32}$$

$$Y_{jik-1/2} = \frac{(1-\delta_{ji})R_{ji}(\tau_{k-1/2})}{1-R_{jj}(\tau_{k-1/2})}, \quad H_{jk-1/2} = \frac{1-\omega(\tau_{k-1/2})}{1-R_{jj}(\tau_{k-1/2})}, \quad k = 1, \ldots, M. \tag{33}$$

is used. Substitute (27) and (20) into (24) and use notation (32) and (33) to obtain the following linear algebraic equations for the intensities $I(\tau_k, u_j)$ ($k = 1, \ldots, M$) in the case of upward radiation ($u_j>0$):

$$I(\tau_{k-1}, u_j)-V_{jk-1/2}I(\tau_k, u_j)-f_{jk-1/2}\sum_{i=0}^{N}Y_{jik-1/2}I(\tau_k, u_i)-W_{jk-1/2}\sum_{i=0}^{N}Y_{jik-1/2}I(\tau_{k-1}, u_i)$$
$$= H_{jk-1/2}(f_{jk-1/2}D_k+W_{jk-1/2}D_{k-1}). \tag{34}$$

Equations (31) and (34), along with boundary conditions (17) and (18), form a closed system of linear algebraic equations for the radiation intensities $I(\tau_k, u_j)$.

Now consider the case when the elements of the matrix $\hat{\mathbf{R}}(\tau)$ vary within the layer $t_{k-1}<\tau<\tau_k$. Then, we substitute (26) and (20) into (23) to obtain, instead of Eqs. (31), the equations

$$I(\tau_k, u_j)-V_{jk-1/2}I(\tau_{k-1}, u_j)-W_{jk-1/2}\sum_{i=0}^{N}Y_{jik}I(\tau_k, u_i)-f_{jk-1/2}\sum_{i=0}^{N}Y_{ijk-1}I(\tau_{k-1}, u_i)$$
$$= W_{jk-1/2}H_{jk}D_k+f_{jk-1/2}H_{jk-1}D_{k-1}, \tag{35}$$

for the intensities of downward radiation ($u_j < 0$) for $k = 1, \ldots, M$, where notation (32) and

$$Y_{jik} = \frac{(1-\delta_{ji})R_{ji}(\tau_k)}{1-R_{jj}(\tau_k)}, \quad H_{jk} = \frac{1-\omega(\tau_k)}{1-R_{jj}(\tau_k)}, \quad k = 0, \ldots, M. \tag{36}$$

are used. Substitute (27) and (20) into (24) and use notation (32) and (36) to obtain, instead of Eqs. (34), the equations

$$\begin{aligned} I(\tau_{k-1}, u_j) - V_{jk-1/2}I(\tau_k, u_j) - f_{jk-1/2}\sum_{i=0}^{N} Y_{jik}I(\tau_k, u_i) - W_{jk-1/2}\sum_{i=0}^{N} Y_{jik-1}I(\tau_{k-1}, u_i) \\ = f_{jk-1/2}H_{jk}D_k + W_{jk-1/2}H_{jk-1}D_{k-1}. \end{aligned} \tag{37}$$

for the intensities of upward radiation ($u_j > 0$) for $k = 1, \ldots, M$. Equations (35) and (37), along with boundary conditions (17) and (18), form a closed system of linear algebraic equations for the radiation intensities $I(\tau_k, u_j)$. Note that the coefficients and the right-hand sides of this system should be calculated using formulas (28), (30), (32), and (36), while the coefficients and the right-hand sides of Eqs. (31) and (34) should be calculated by formulas (28), (29), (32), and (33).

5. SOLUTIONS OF THE EQUATIONS BY THE BLOCK ELIMINATION METHOD

Each of the two systems of linear equations for the intensities $I(\tau_k, u_j)$ can be represented by a system of three-point vector equations. To this end, we introduce the column vectors of radiation intensities at the grid nodes u_i at the altitudes z_{M-k}

$$\mathbf{I}_k = (I(\tau_k, u_0), \ldots, I(\tau_k, u_N))^{\mathrm{T}}, \quad k = 0, \ldots, M.$$

The system of equations (31), (34) along with boundary conditions (17, (18) and system (35), (37) along with boundary conditions (17, (18) can be written as the following system of linear three-point vector equations for the vectors $\mathbf{I}_k$:

$$\begin{aligned} \hat{\mathbf{C}}_0\mathbf{I}_0 - \hat{\mathbf{B}}_0\mathbf{I}_1 &= \mathbf{F}_0, \\ -\hat{\mathbf{A}}_k\mathbf{I}_{k-1} + \hat{\mathbf{C}}_k\mathbf{I}_k - \hat{\mathbf{B}}_k\mathbf{I}_{k+1} &= \mathbf{F}_k, \quad k = 1, \ldots, M-1, \\ -\hat{\mathbf{A}}_M\mathbf{I}_{M-1} + \hat{\mathbf{C}}_M\mathbf{I}_M &= \mathbf{F}_M. \end{aligned} \tag{38}$$

In these equations, $\hat{\mathbf{A}}_k$, $\hat{\mathbf{B}}_k$, and $\hat{\mathbf{C}}_k$ are square matrices of size $(N+1) \times (N+1)$ and $\mathbf{F}_k$ are column vectors of size $N+1$. The components of these matrices and vectors are denoted by A_{jik}, B_{jik}, C_{jik}, and F_{jk}, respectively.

Consider system (31), (34) obtained under the assumption that the single-scattering albedo and the scattering indicatrix are constant within each layer $\tau_{k-1} < \tau < \tau_k$. Then, the components of the matrices $\hat{\mathbf{A}}_k$, $\hat{\mathbf{B}}_k$, $\hat{\mathbf{C}}_k$, and vectors $\mathbf{F}_k$ for $k = 1, \ldots, M-1$ are given by the formulas

$$A_{jik} = \begin{cases} \delta_{ji}V_{jk-1/2} + f_{jk-1/2}Y_{jik-1/2}, & j = 0, \ldots, (N-1)/2, \quad i = 0, \ldots, N, \\ 0, \quad j = (N+1)/2, \ldots, N, & i = 0, \ldots, N, \end{cases}$$

$$B_{jik} = \begin{cases} 0, \quad j = 0, \ldots, (N-1)/2, & i = 0, \ldots, N, \\ \delta_{ji}V_{jk+1/2} + f_{jk+1/2}Y_{jik+1/2}, & j = (N+1)/2, \ldots, N, \quad i = 0, \ldots, N, \end{cases}$$

$$C_{jik} = \begin{cases} \delta_{ji} - W_{jk-1/2}Y_{jik-1/2}, & j = 0, \ldots, (N-1)/2, \quad i = 0, \ldots, N, \\ \delta_{ji} - W_{jk+1/2}Y_{jik+1/2}, & j = (N+1)/2, \ldots, N, \quad i = 0, \ldots, N, \end{cases}$$

$$F_{jk} = \begin{cases} H_{jk-1/2}(W_{jk-1/2}D_k + f_{jk-1/2}D_{k-1}), & j = 0, \ldots, (N-1)/2, \\ H_{jk+1/2}(f_{jk+1/2}D_{k+1} + W_{jk+1/2}D_k), & j = (N+1)/2, \ldots, N. \end{cases}$$

The matrices $\hat{\mathbf{A}}_M$, $\hat{\mathbf{C}}_M$, and the column vector $\mathbf{F}_M$ are determined by the formulas

$$C_{jiM} = \begin{cases} \delta_{ji} - W_{jM-1/2}Y_{jiM-1/2}, & j = 0, \dots, (N-1)/2, \quad i = 0, \dots, N, \\ -\Omega(\nu)\dfrac{u_1 - u_0}{2}, & j = (N+1)/2, \dots, N, \quad i = 0, \\ -\Omega(\nu)\dfrac{u_{i+1} - u_{i-1}}{2}, & j = (N+1)/2, \dots, N, \quad i = 1, \dots, (N-1)/2, \\ \delta_{ji}, & j = (N+1)/2, \dots, N, \quad i = (N+1)/2, \dots, N, \end{cases}$$

$$A_{jiM} = \begin{cases} \delta_{ji}V_{jM-1/2} + f_{jM-1/2}Y_{jiM-1/2}, & j = 0, \dots, (N-1)/2, \quad i = 0, \dots, N, \\ 0, & j = (N+1)/2, \dots, N, \quad i = 0, \dots, N, \end{cases}$$

$$F_{jM} = \begin{cases} H_{jM-1/2}(W_{jM-1/2}D_M + f_{jM-1/2}D_{M-1}), & j = 0, \dots, (N-1)/2, \\ (1-\Omega(\nu))B(T_p, \nu), & j = (N+1)/2, \dots, N. \end{cases}$$

The matrices $\hat{\mathbf{B}}_0$, $\hat{\mathbf{C}}_0$, and the column vector $\mathbf{F}_0$ are determined by the formulas

$$B_{ji0} = \begin{cases} 0, & j = 0, \dots, (N-1)/2, \quad i = 0, \dots, N, \\ \delta_{ji}V_{j1/2} + f_{j1/2}Y_{ji1/2}, & j = (N+1)/2, \dots, N, \quad i = 0, \dots, N, \end{cases}$$

$$C_{ji0} = \begin{cases} \delta_{ji}, & j = 0, \dots, (N-1)/2, \quad i = 0, \dots, N, \\ \delta_{ji} - W_{j1/2}Y_{ji1/2}, & j = (N+1)/2, \dots, N, \quad i = 0, \dots, N, \end{cases}$$

$$F_{j0} = \begin{cases} 0, & j = 0, \dots, (N-1)/2, \\ H_{j1/2}(f_{j1/2}D_1 + W_{j1/2}D_0), & j = (N+1)/2, \dots, N. \end{cases}$$

Now consider system (35), (37) obtained under the assumption that the single-scattering albedo and the scattering indicatrix vary within each layer $\tau_{k-1} < \tau < \tau_k$. Then, the components of the matrices $\hat{\mathbf{A}}_k$, $\hat{\mathbf{B}}_k$, $\hat{\mathbf{C}}_k$, and vectors $\mathbf{F}_k$ for $k = 1, \dots, M-1$ are given by the formulas

$$A_{jik} = \begin{cases} \delta_{ji}V_{jk-1/2} + f_{jk-1/2}Y_{jik-1}, & j = 0, \dots, (N-1)/2, \quad i = 0, \dots, N, \\ 0, & j = (N+1)/2, \dots, N, \quad i = 0, \dots, N, \end{cases}$$

$$B_{jik} = \begin{cases} 0, & j = 0, \dots, (N-1)/2, \quad i = 0, \dots, N, \\ \delta_{ji}V_{jk+1/2} + f_{jk+1/2}Y_{jik+1}, & j = (N+1)/2, \dots, N, \quad i = 0, \dots, N, \end{cases}$$

$$C_{jik} = \begin{cases} \delta_{ji} - W_{jk-1/2}Y_{jik}, & j = 0, \dots, (N-1)/2, \quad i = 0, \dots, N, \\ \delta_{ji} - W_{jk+1/2}Y_{jik}, & j = (N+1)/2, \dots, N, \quad i = 0, \dots, N, \end{cases}$$

$$F_{jk} = \begin{cases} W_{jk-1/2}H_{jk}D_k + f_{jk-1/2}H_{jk-1}D_{k-1}, & j = 0, \dots, (N-1)/2, \\ f_{jk+1/2}H_{jk+1}D_{k+1} + W_{jk+1/2}H_{jk}D_k, & j = (N+1)/2, \dots, N. \end{cases}$$

The matrices $\hat{\mathbf{A}}_M$, $\hat{\mathbf{C}}_M$, and the column vector $\mathbf{F}_M$ are determined by the formulas

$$C_{jiM} = \begin{cases} \delta_{ji} - W_{jM-1/2}Y_{jiM}, & j = 0, \dots, (N-1)/2, \quad i = 0, \dots, N, \\ -\Omega(\nu)\dfrac{u_1 - u_0}{2}, & j = (N+1)/2, \dots, N, \quad i = 0, \\ -\Omega(\nu)\dfrac{u_{i+1} - u_{i-1}}{2}, & j = (N+1)/2, \dots, N, \quad i = 1, \dots, (N-1)/2, \\ \delta_{ji}, & j = (N+1)/2, \dots, N, \quad i = (N+1)/2, \dots, N, \end{cases}$$

$$A_{jiM} = \begin{cases} \delta_{ji}V_{jM-1/2} + f_{jM-1/2}Y_{jiM-1}, & j = 0, \dots, (N-1)/2, \quad i = 0, \dots, N, \\ 0, & j = (N+1)/2, \dots, N, \quad i = 0, \dots, N, \end{cases}$$

$$F_{jM} = \begin{cases} W_{jM-1/2}H_{jM}D_M + f_{jM-1/2}H_{jM-1}D_{M-1}, & j = 0, \dots, (N-1)/2, \\ (1 - \Omega(\nu))B(T_p, \nu), & j = (N+1)/2, \dots, N. \end{cases}$$

The matrices $\hat{\mathbf{B}}_0$, $\hat{\mathbf{C}}_0$, and the column vector $\mathbf{F}_0$ are determined by the formulas

$$B_{ji0} = \begin{cases} 0, & j = 0, \dots, (N-1)/2, \quad i = 0, \dots, N, \\ \delta_{ji}V_{j1/2} + f_{j1/2}Y_{ji1}, & j = (N+1)/2, \dots, N, \quad i = 0, \dots, N, \end{cases}$$

$$C_{ji0} = \begin{cases} \delta_{ji}, & j = 0, \dots, (N-1)/2, \quad i = 0, \dots, N, \\ \delta_{ji} - W_{j1/2}Y_{ji0}, & j = (N+1)/2, \dots, N, \quad i = 0, \dots, N, \end{cases}$$

$$F_{j0} = \begin{cases} 0, & j = 0, \dots, (N-1)/2, \\ f_{j1/2}H_{j1}D_1 + W_{j1/2}H_{j0}D_0, & j = (N+1)/2, \dots, N. \end{cases}$$

The system of linear three-point vector equations (38) is solved by the block elimination method described in [9]. This method operates in two phases. In the first phase, the first pass of block elimination is performed. In this pass, the matrix $\hat{\mathbf{G}}_1$ and the column vector $\mathbf{D}_1$ are first computed using the formulas

$$\hat{\mathbf{G}}_1 = \hat{\mathbf{C}}_0^{-1}\hat{\mathbf{B}}_0, \quad \mathbf{D}_1 = \hat{\mathbf{C}}_0^{-1}\mathbf{F}_0.$$

Next, the matrices $\hat{\mathbf{G}}_{k+1}$ and the column vectors $\mathbf{D}_{k+1}$ for $k = 1, \dots, M-1$ are successively computed by the formulas

$$\hat{\mathbf{G}}_{k+1} = (\hat{\mathbf{C}}_k - \hat{\mathbf{A}}_k\hat{\mathbf{G}}_k)^{-1}\hat{\mathbf{B}}_k, \quad \mathbf{D}_{k+1} = (\hat{\mathbf{C}}_k - \hat{\mathbf{A}}_k\hat{\mathbf{G}}_k)^{-1}(\mathbf{F}_k + \hat{\mathbf{A}}_k\mathbf{D}_k).$$

Next, the second pass of block elimination is performed. First, the column vector $\mathbf{I}_M$, and then the column vectors $\mathbf{I}_k$ for $k = M-1, \dots, 0$ are successively computed by the formulas

$$\mathbf{I}_M = (\hat{\mathbf{C}}_M - \hat{\mathbf{A}}_M\hat{\mathbf{G}}_M)^{-1}(\mathbf{F}_M + \hat{\mathbf{A}}_M\mathbf{D}_M), \quad \mathbf{I}_k = \hat{\mathbf{G}}_{k+1}\mathbf{I}_{k+1} + \mathbf{D}_{k+1}.$$

The block elimination method is stable if the matrix norms (arbitrary norms equivalent to the Euclidean one) satisfy the conditions

$$\left\|\hat{\mathbf{C}}_0^{-1}\hat{\mathbf{B}}_0\right\| \le 1, \quad \left\|\hat{\mathbf{C}}_M^{-1}\hat{\mathbf{A}}_M\right\| \le 1, \quad \left\|\hat{\mathbf{C}}_k^{-1}\hat{\mathbf{A}}_k\right\| + \left\|\hat{\mathbf{C}}_k^{-1}\hat{\mathbf{B}}_k\right\| \le 1, \quad k = 1, \dots, M-1.$$

Numerical computations showed that the stability condition is always satisfied even if the single-scattering albedo is very close to unity. These computations also showed that the Gauss–Jordan matrix inversion method with complete pivoting with respect to columns requires double precision computations.

6. SOFTWARE IMPLEMENTATION OF THE METHOD AND NUMERICAL RESULTS

Both versions of the method for the calculation of the transfer of monochromatic radiation in the horizontally homogeneous atmosphere described above were implemented in computer programs. These programs are organized in the same way as in DISORT. The input data for the program in the case when the single-scattering and the scattering indicatrix are constant within each layer consist of the single-scattering albedo, the surface temperature, the zenith angle grid, the optical thickness grid beginning from the upper boundary, the single-scattering albedo values and the atmospheric gas temperature at the nodes of the latter grid, and the coefficients of the expansion of the scattering indicatrix in Legendre polynomials. In the case when the single-scattering albedo and the scattering indicatrix vary within each layer, the extinction coefficient at the nodes of the optical thickness grid are also provided in the input data. The matrices were inverted by the Gauss–Jordan method with complete pivoting with respect to columns. All the computations are performed in double precision.

We calculated the field of intrinsic intensity of the Venusian atmosphere in the altitude range from 0 through 100 km for a wave length in the near infrared range using the program developed within this study and the DISORT program in order to compare the accuracy and speed of both programs. The altitude grid step was 250 m. The number of grid points in the zenith angle grid in both programs was identical between 16 and 42 points. In the program developed within the present study, a uniform zenith angle grid was used. In DISORT, due to the use of Gauss–Legendre quadrature formulas for the approximation of integrals over the scattering angles, the zenith angle grid points are roots of the Legendre polynomials with the indices by one greater than the number of points in the grid. All the input data, except for the uniform zenith angle grid points, were identical for both programs. The parameters of the Venusian atmosphere were determined by the standard VIRA model (see [10]). An important feature of the Venusian atmosphere is that it contains at the altitudes of 50–70 km a cloud layer with a large optical thickness in the near IR range in which the single-scattering albedo is very close to unity (greater than 0.9996) and the scattering indicatrix is strongly elongated to the front.

Figure 1 shows the profiles of extinction coefficients, the single-scattering albedo, and the asymmetry parameter of the scattering indicatrix (see [1–3]) for the radiation at the wave length of 1.25 μm in the Venusian atmosphere that were used in the computations. It is seen that the single-scattering albedo above 48 km is close to unity, which indicates significant predominance of scattering over absorption. It is also seen that the scattering indicatrix is strongly elongated to the front at the altitudes of 49–85 km.

Figure 2 illustrates the dependence of the decimal logarithm of the intensity of intrinsic radiation with the wave length of 1.25 μm in the Venusian atmosphere at the altitude and zenith angle. These dependences were computed using the program described in this paper on the zenith angle grid consisting of 40 nodes. It is seen from Fig. 2 that the intensity of radiation is considerably different at different altitudes. Below 10 km, it is almost independent of the zenith angle. At higher altitudes, the dependence becomes stronger up to the lower boundary of the cloud layer at 48 km. At the altitudes of 50–55 km, the intensity of radiation in the cloud layer is almost independent of the zenith angle due to strong scattering. The dependence on the zenith angle becomes strong as the altitude increases further. At the altitude of 80 km, the intensity of the upward radiation exceeds the intensity of the downward radiation by a factor greater than 1000. At the altitudes of 95–100 km, these intensities differ by more than eight orders of magnitude.

Figure 3 illustrates the dependence on the altitude and zenith angle of the relative difference between the intensity of radiation with the wave length of 1.25 μm in the Venusian atmosphere computed using DISORT on the zenith grid consisting of 40 nodes and the same intensity computed by the proposed program (the latter intensity is shown in Fig. 2). The computed intensity of radiation was interpolated to the uniform zenith angle grid. The relative difference was calculated by the formula $2|I_D - I|/(I_D + I)$ in which I_D is the intensity of radiation computed by DISORT and I is the intensity of radiation computed by the program described in this paper. It is seen in Fig. 3 that the intensities computed by the two programs differ at all altitudes not greater than 4.5%. Moreover, the relative difference greater than 1% is observed only at the zenith angles of 80–110 degrees at the altitudes of 30–48 km and at the altitudes of 80–100 km, where the intensity of radiation varies by 4–5 orders of magnitude when the zenith angle cosine varies from –0.1 to 0.1. It is seen that at the altitudes of 0–36 and 48–80 km the relative difference does not exceed 0.5%.

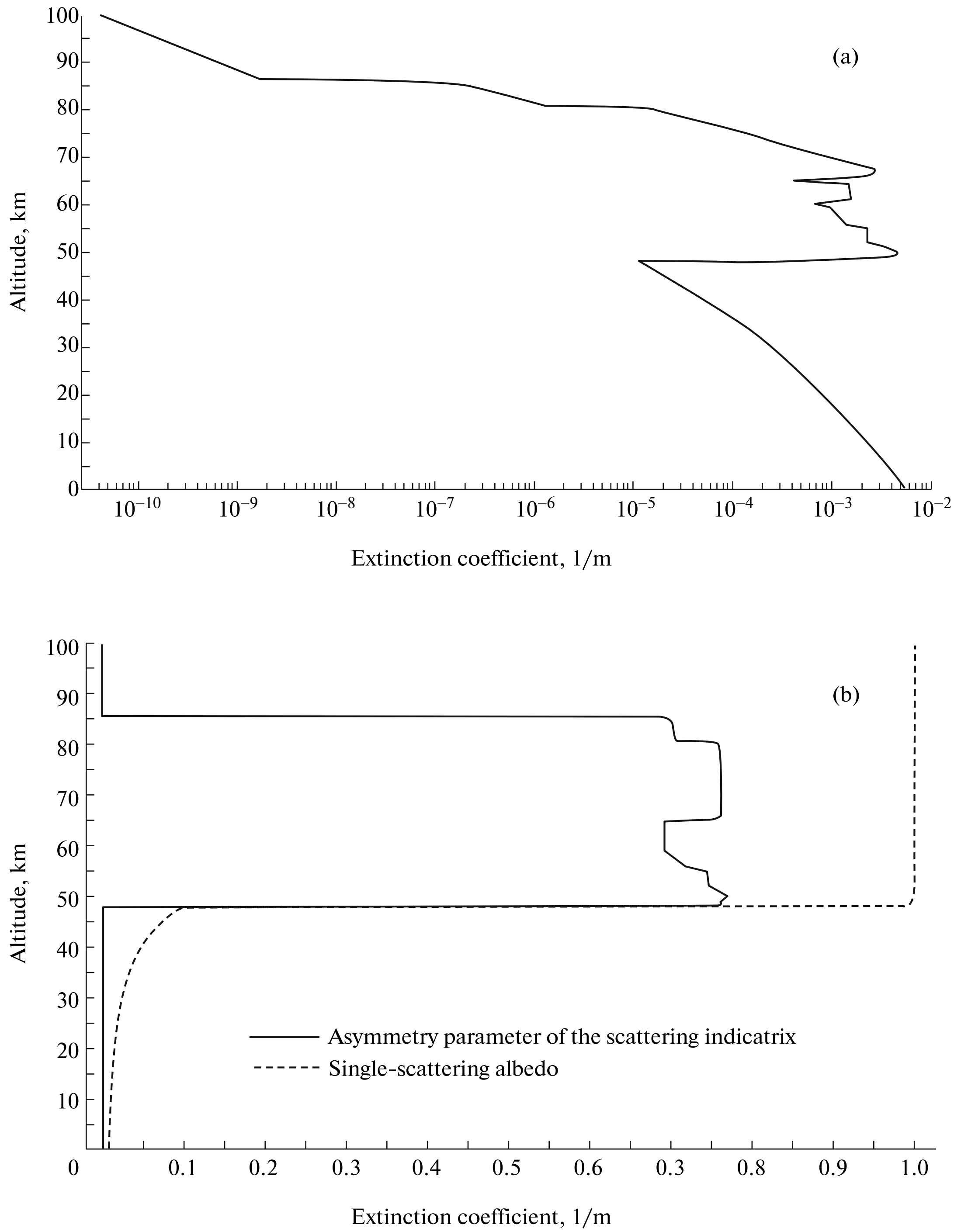

Fig. 1. The vertical profiles of extinction coefficient (a), the single-scattering albedo, and the asymmetry parameter of the scattering indicatrix (b) for the radiation at the wave length of 1.25 μm in the Venusian atmosphere.

The computations showed that when identical optical thickness and zenith angle grids with the same number of grid nodes are used by both programs, the new program described in the present paper and DISORT yield very close results. Differences in the integral upward and downward radiation fluxes computed by both programs did not exceed 1% at all the altitudes. The computations also showed that the new program is approximately twice as fast as DISORT. In addition, it turned out that, for the given number of zenith angle grid nodes, the new program yields the most accurate results when the zenith angle grid is uniform.

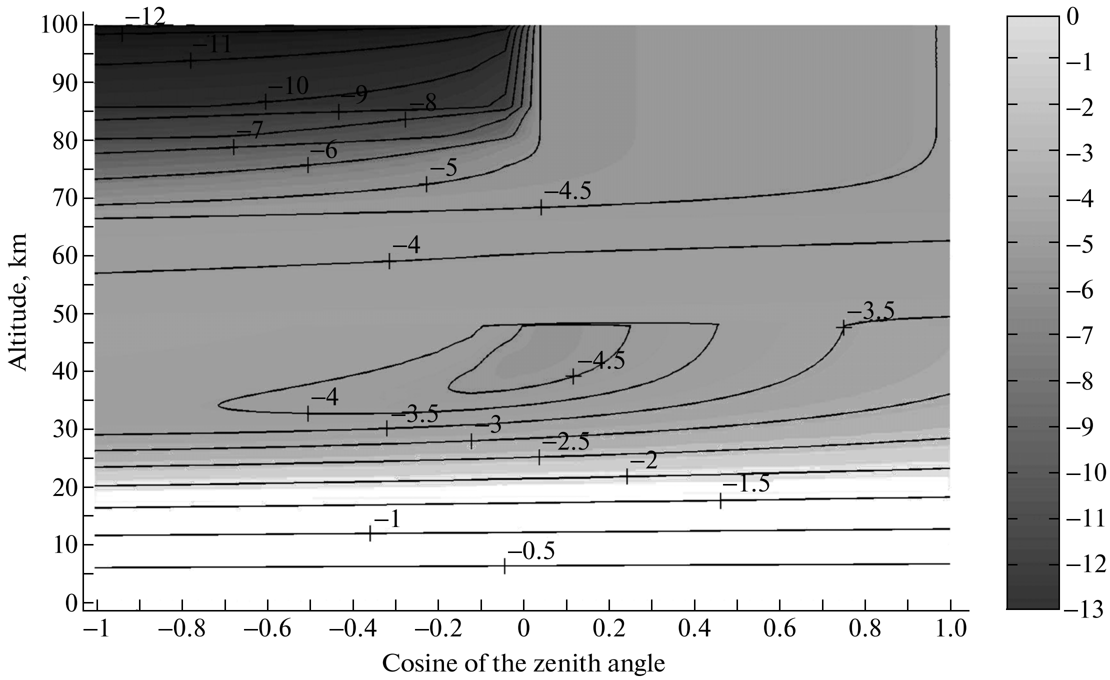

Fig. 2. The dependence of the decimal logarithm of the intensity of intrinsic radiation at the wave length of 1.25 μm in the Venusian atmosphere on the altitude and zenith angle in ergs (s cm^2 cm^{-1})$^{-1}$. These dependence was computed by the program proposed in the present paper.

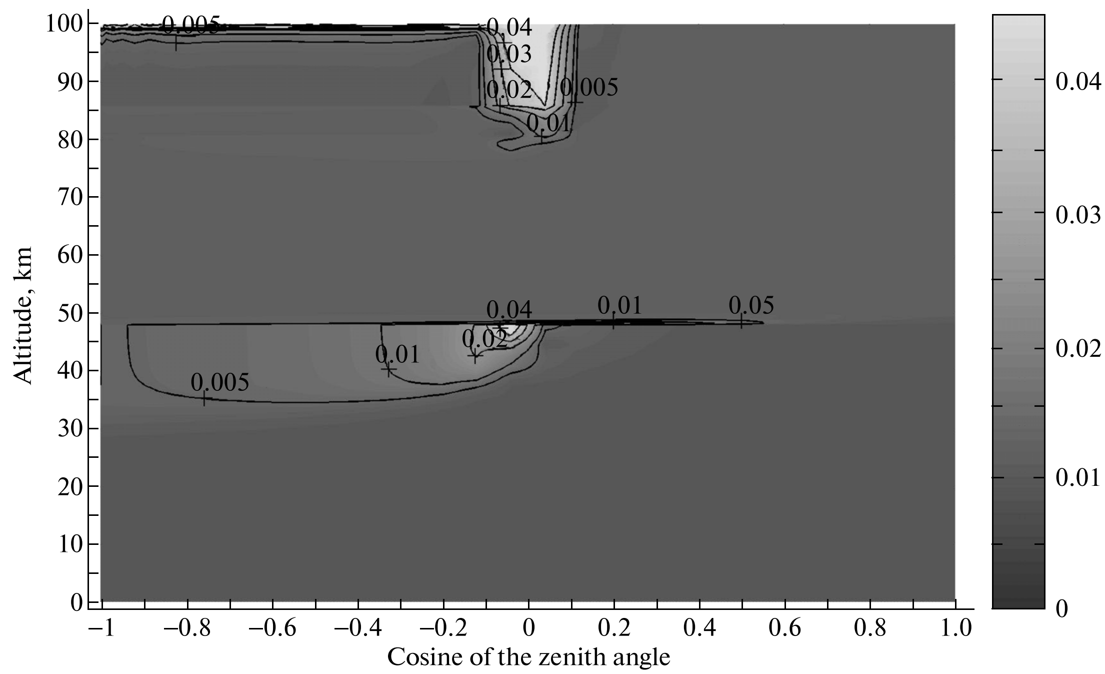

Fig. 3. The dependence on the altitude and zenith angle of the relative difference between the intensity of intrinsic radiation with the wave length of 1.25 μm in the Venusian atmosphere computed using DISORT and the same intensity computed by the program proposed in the present paper.

The comparison of two versions of the method proposed in this paper showed that in the case of small altitude grid step (not greater than 250 m), both versions yield almost identical results. For greater grid steps (500 m and greater), the version in which the single-scattering albedo and scattering indicatrix vary within each layer is more accurate.

7. CONCLUSIONS

A new version of the discrete ordinate method for the calculation of the transfer of monochromatic radiation in a scattering, absorbing, and emitting plane-parallel atmosphere is proposed. The implementation of this method is relatively simple, and it has three specific features. The first feature is that the proposed method analytically takes into account the fact that a part of scattered radiation preserves the zenith angle. The second feature is that an arbitrary zenith angle grid may be used (the Gauss–Legendre quadrature formulas are not used). The third feature is that the system of linear algebraic equations produced by the discrete ordinate method is solved by the block elimination method. This is an exact and computationally efficient method, which uses the structure of the coefficient matrix to reduce the amount of computations. Typically, this system was solved by iterative methods or by calculating the eigenvalues and eigenvectors of the large-scale coefficient matrix of the linear system. The implementation of these methods is more difficult and they are less efficient than the block elimination method. Numerical computations showed that the proposed method is stable and accurate, and it is about two times faster than the program used in the free DISORT package.

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