Stabilization by Unilateral Controls

and

Variational Analysis

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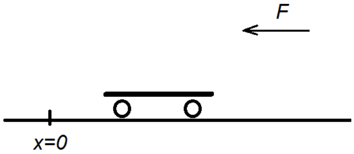
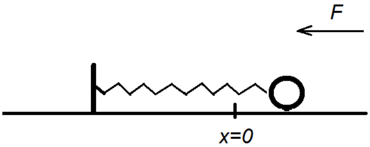
Unilateral Control

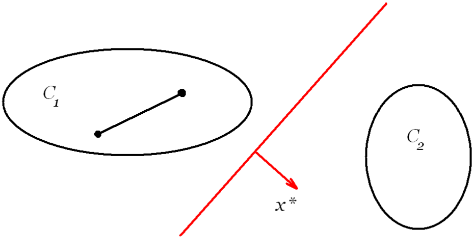




Euler, 1744

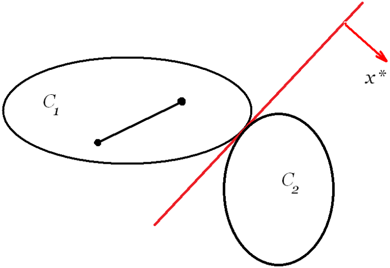
Quum enim Mundi Universi fabrica sit perfectissima, atque a Creatore sapientissimo absoluta, nihil omnino in Mundo contingit in quo non Maximi Minimive ratio quaepiam eluceat; quamobrem dubium prorsus est nullum quim omnes Mundi Methodi Maximorum et Minimorum.

Separation Theorem, Minkowski, 1896



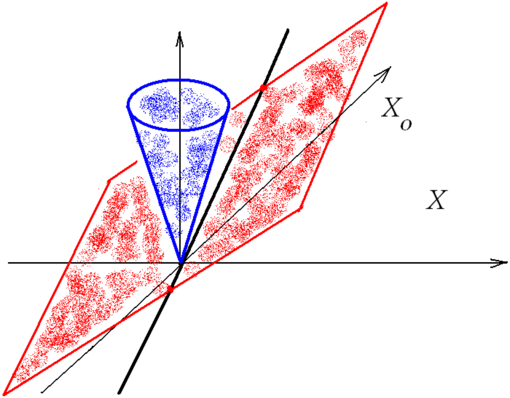
$$\langle x^*, x_1 \rangle \leq \langle x^*, x_2 \rangle, \ \forall\, x_1 \in C_1,\, x_2 \in C_2.$$

Separation Theorem → Variational Analysis

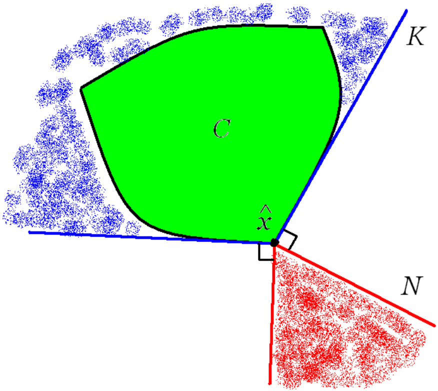


$$\langle x^*, x_1 \rangle \le \langle x^*, x_2 \rangle, \ \forall\, x_1 \in C_1,\ x_2 \in C_2.$$

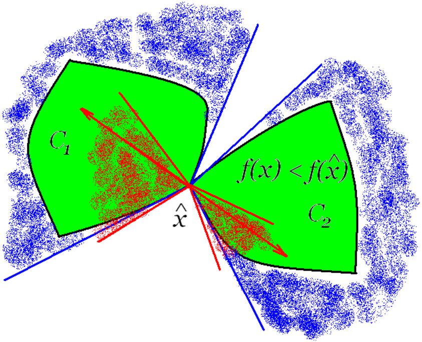
Hahn-Banach Theorem, 1929



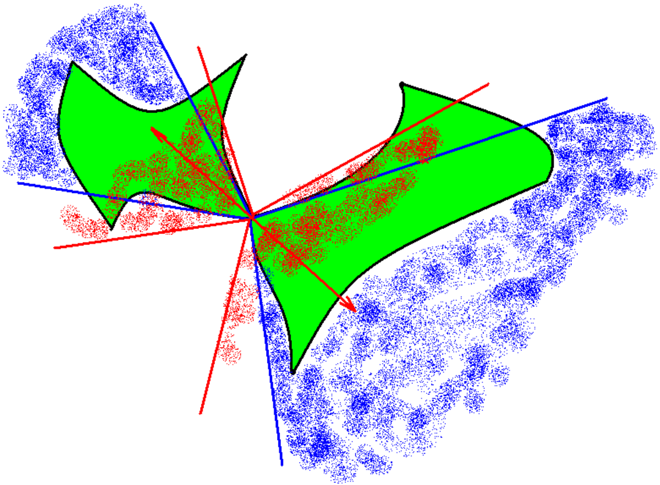
Tangent and Normal Cones



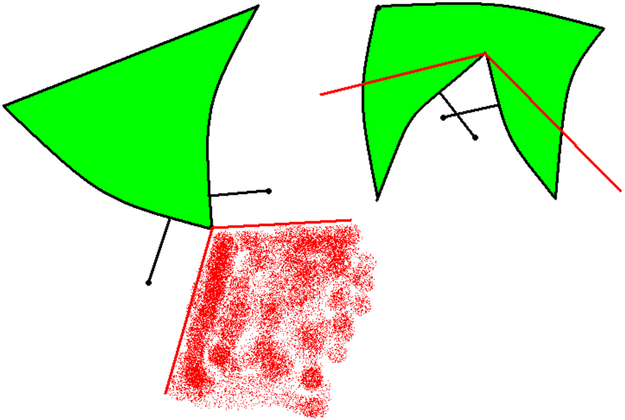
Karush-Kuhn-Tucker Theorem, 1951



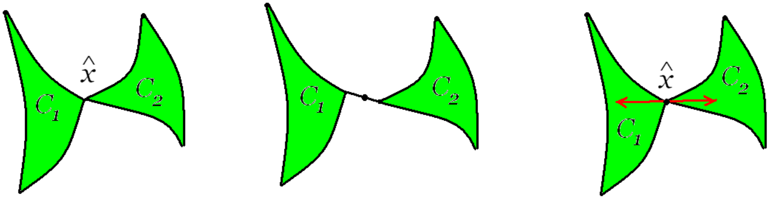
Lagrange Multiplier Rule, Clarke, 1975



Mordukhovich Normal Cone, 1976

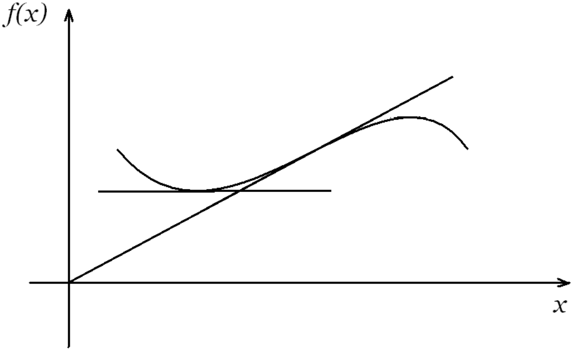


Separation Theorem: Kruger-Mordukhovich, 1980

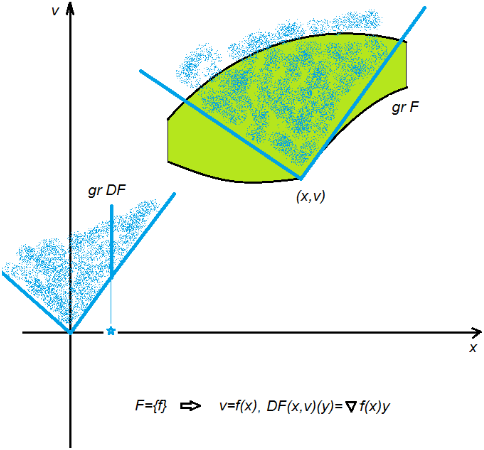


$$0 \in N(\hat{x}, C_1) + N(\hat{x}, C_2).$$

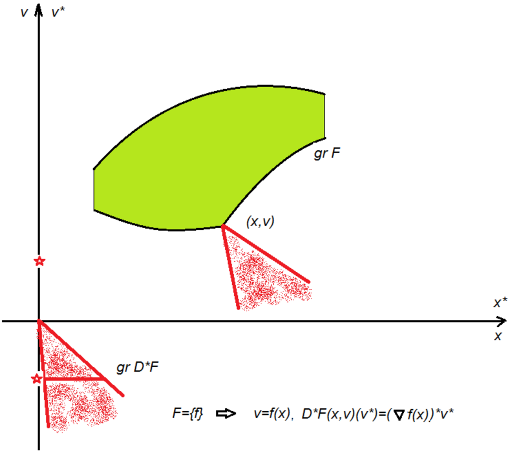
Derivative: Fermat, 1629



Set-Valued Map Derivative



Set-Valued Map Co-Derivative



Stabilization problem

$$\dot{x} = f(x,u),\ \ u \in U,\ \ 0 = f(0,u_0)$$

The problem is to find a function $u(x) \in U$ satisfying the following conditions:

1. $u(0) = u_0$,
2. $x = 0$ is an asymptotically stable equilibrium position of the closed loop system
$$\dot{x} = f(x,u(x)).$$

Stabilization problem in terms of differential inclusions

$$\text{If } U = U(x) \Rightarrow \dot{x} \in F(x) = f(x, U(x))$$

$$\dot{x} \in F(x), \;\; 0 \in F(0)$$

The problem is to find a function $f(x) \in F(x)$ satisfying the following conditions:

1. $f(0) = 0$,
2. $x = 0$ is an asymptotically stable equilibrium position of the system

$$\dot{x} = f(x).$$

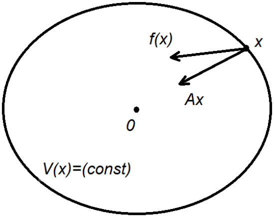
Stabilization and weak asymptotic stability (asymptotic controllability)

$$\dot{x} \in F(x), \;\; 0 \in F(0)$$

$$\forall\, x_0 \;\; \exists\, x(t, x_0) \to 0,\; t \to +\infty \;?$$

Stability via the Lyapunov first method

$$\dot{x} = f(x),\ \ 0 = f(0),\ \ \dot{x} = Ax,\ \ A = \nabla f(0)$$



First approximation for differential inclusions: convex process

$$\dot{x} \in F(x), \;\; 0 \in F(0), \;\; \dot{x} \in A(x), \;\; \mathrm{grA} \subset \mathrm{gr}DF(0,0)$$

$$A(x_1 + x_2) \supset A(x_1) + A(x_2) \;\; \text{(convex process)}$$

$$\mathrm{dom}A = \{x \in R^n \mid A(x) \neq \emptyset\} = R^n$$

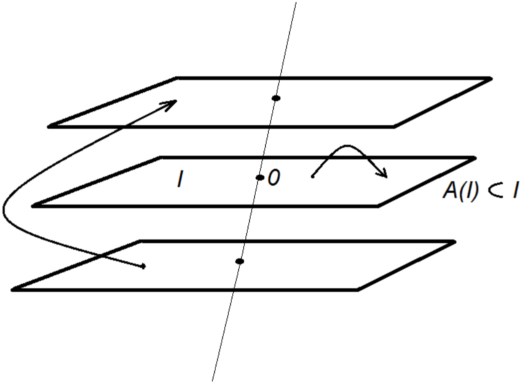
$$A^*(v^*) = \{x^* \mid \langle x^*, x\rangle \leq \langle v^*, v\rangle, \; \forall (x,v) \in \mathrm{gr}A\} \supset D^*F(0,0)(v^*)$$

Example:

$$F(x) = f(x,U), \;\; A(x) = \nabla_x f(0,u_0)x + \mathrm{cone}f(0,U)$$

$$A^*(v^*) = \begin{cases} (\nabla f_x(0,u_0))^* v^*, & -v^* \in N(0, f(0,U)), \\ \emptyset, & -v^* \notin N(0, f(0,U)). \end{cases}$$

Structure of a convex process: generalized Jordan theorem



Structure of a convex process

Consider convex cones

$$L_k(\lambda) = (A - \lambda I)^{-k}(0), \;\; k = 1, 2, \dots$$

and put

$$L(\lambda) = \bigcup_{k \geq 1} L_k(\lambda).$$

The set of eigenvalues of A^* is bounded and closed. Denote by $\lambda_0(A^*)$ the maximal eigenvalue of the process A^*. If A^* has no an eigenvector, put $\lambda_0(A^*) = -\infty$.

Theorem

If $\lambda > \lambda_0(A^*)$, then $L(\lambda) = I$.

Stabilization

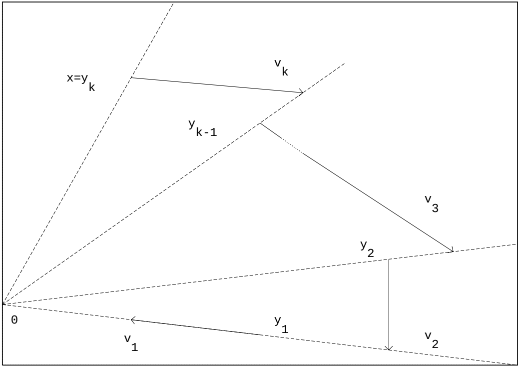
If $x \in I$, then there exists a finite collection of vectors $y_i \in I$, $i = \overline{1,k}$, such that

$$\begin{array}{rcl} \lambda y_1 & \in & A(y_1), \\ y_1 + \lambda y_2 & \in & A(y_2), \\ *** \;\; *** & & *** \\ y_{k-1} + \lambda y_k & \in & A(y_k), \\ y_k & = & x. \end{array} \tag{1}$$

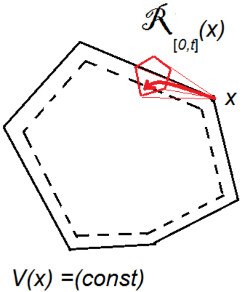
If $\lambda < 0$ we have

$$\frac{1}{|\lambda|} y_{k-1} \in y_k + \frac{1}{|\lambda|} A(y_k).$$

Stabilization

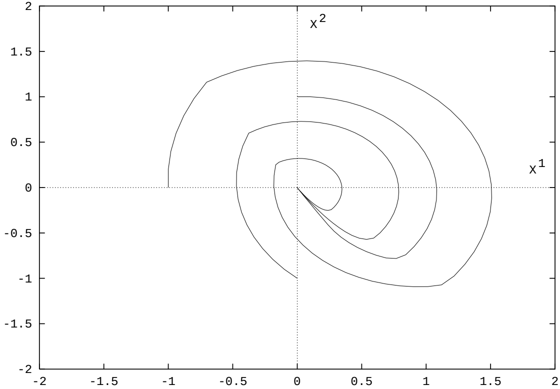


How it works? Reachability sets

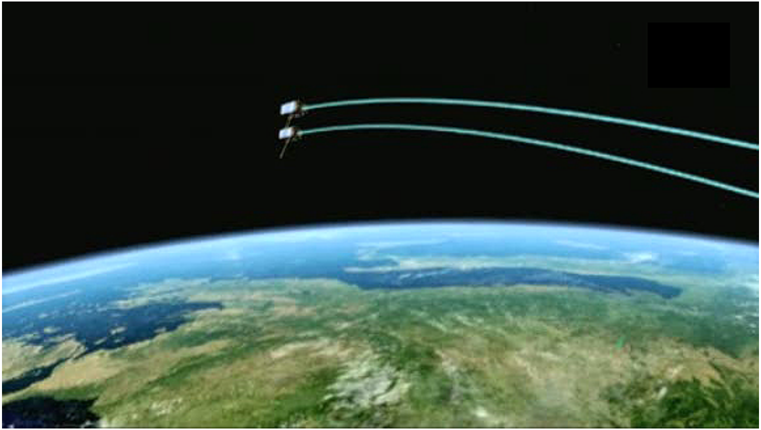


A.V. Lotov (FAPM - 1969)
and his School

Oscillator with a unilateral control



Formation flying



Equations of motion

Nonlinear equatuions:

$$\ddot{r}_1 = g(r_1) + J_2(r_1),$$
$$\ddot{r}_2 = g(r_2) + J_2(r_2) + h(r_2)u,$$
$$u \in R \text{ or } u \geq 0.$$

Schweighart-Sedwick equations of relative motion with a single-input control:

$$\ddot{x} + 2nc\dot{z} = w(t)e_x(t),$$
$$\ddot{y} + q^2 y = 2lq\cos(qt + \phi) + w(t)e_y(t),$$
$$\ddot{z} - 2nc\dot{x} - (5c^2 - 2)n^2 z = w(t)e_z(t).$$

Control oriented along the geomagnetic field

$$e_x(t) = \frac{\cos\theta(t)\sin i_2}{\sqrt{1+3\sin^2\theta(t)\sin^2 i_2}},$$

$$e_y(t) = \frac{\cos i_2}{\sqrt{1+3\sin^2\theta(t)\sin^2 i_2}},$$

$$e_z(t) = \frac{-2\sin\theta(t)\sin i_2}{\sqrt{1+3\sin^2\theta(t)\sin^2 i_2}},$$

$$\theta(t) \approx nct.$$

Theorem

If $\sin 2i_2 \neq 0$, then the Schweighart-Sedwick system is 4π-controllable.

Newton's method

$$f(x) = 0,$$

$$x^{k+1} = x^k + \bar{x}^k, \ \ k = 0, 1, \ldots,$$

where $\bar{x}^k$ is a solution to the system of linear equations

$$\nabla f(x^k)\bar{x}^k = -f(x^k).$$

$$0 \in F(x),$$

$$-v^k \in \Lambda(x^k, v^k)(\bar{x}^k),$$

where v^k is a nearest to zero point belonging to the set $F(x^k)$.

$$\mathrm{gr}\Lambda(x^k, v^k) \subset DF(x^k, v^k)$$

Metric regularity

The condition of non-singularity of the matrix $\nabla f(x^k)$, essential to solve linear system and to prove convergence theorems, is substituted by the condition of metric regularity. A set-valued map $F : R^n \to R^m$ is metrically regular around $(\hat{x}, \hat{v}) \in \text{gph} F$ if there exists $\varepsilon > 0$ as well as a number $\mu > 0$ such that

$$d(x, F^{-1}(v)) \leq \mu d(v, F(x)), \;\; x \in \hat{x} + \varepsilon B_n, \;\; v \in \hat{v} + \varepsilon B_m.$$

Theorem (Mordukhovich criterion)

A set-valued map $F : R^n \to R^m$ with closed graph is metrically regular around $(\hat{x}, \hat{v}) \in \text{gph} F$ if and only if the inclusion $0 \in D^* F(\hat{x}, \hat{v})(v^*)$ implies that $v^* = 0$.

Newton's method and metric regularity

We say that the Newton method for perturbed set-valued map F is well-posed around $(\hat{x},\hat{v}) \in \mathrm{gph}F$ with modulus μ if there exists $\eta > 0$ such that for all $x \in \hat{x} + \eta B_n$ and $\tilde{v} \in \hat{v} + \eta B_m$ there is $v \in \pi(0, F(x) - \tilde{v})$ satisfying the condition

$$DF^{-1}(v+\tilde{v},x)(-v) \cap \mu\|v\|B_n \neq \emptyset.$$

This condition implies that the largest possible Newton inclusion for the perturbed map $x \to F(x) - \tilde{v}$ (the generalization of Newton's equation) has at least one solution, $\bar{x}$, satisfying the boundedness condition $\bar{x} \in \mu B_n$.

Theorem

Assume that the set-valued map $F : R^n \to R^m$ with closed values is Lipschitzian with the constant $L_F > 0$ (i.e. $F(x_1) \subset F(x_2) + L_F\|x_1 - x_2\|B_n$) in a neighbourhood of a point $\hat{x}$. Then the following conditions are equivalent:

1. *The map F is metrically regular around $(\hat{x},\hat{v}) \in \mathrm{gph}F$.*
2. *Newton's method for perturbed set-valued map F is well-posed around $(\hat{x},\hat{v}) \in \mathrm{gph}F$.*

Newton's method

$$f(x,u) = 0, \ \ u \in U.$$

$$K = \{u \in R^l \mid u + U \subset U\}$$

Newton's inclusion:

$$-v \in \Lambda(x,u)(\bar{x}) = \nabla_x f(x,u)\bar{x} + \nabla_u f(x,u)K.$$

Newton's method is well defined if

$$((\nabla_u f(\hat{x},\hat{u}))^T)^{-1} K^* \cap \ker(\nabla_x f(\hat{x},\hat{u}))^T = \{0\}.$$

Application to formation flying

Relative motion of 2 satellites:

$$\dot{z} = \phi(t, z, u), \;\; z \in R^6, \;\; u \in \mathcal{U} = R_+ = \{u \mid u \geq 0\}, \;\; t \in [0, T], \tag{2}$$

Discrete-time system

$$z_{k+1} = z_k + \tau\phi(k\tau, z_k, u_k), \;\; u_k \in \mathcal{U}, \;\; k = \overline{0, N-1}. \tag{3}$$

Here $\tau = T/N$.

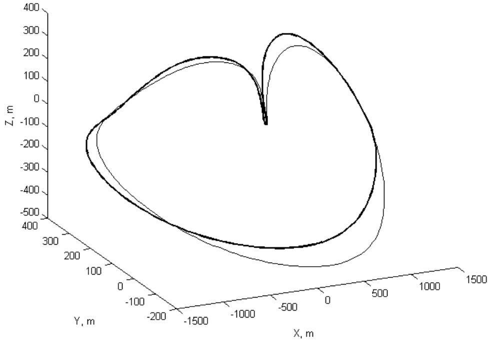
Application to formation flying

$$F(Z,U)=0,\ \ U\in\mathcal{U}^N,$$
$$F: R^{6(N-1)}\times\mathcal{U}^N\to R^{6N}$$

$$F(Z,U)=\left(\begin{array}{c} z_1-z_0-\tau\phi(0,z_0,u_0)\\ z_2-z_1-\tau\phi(\tau,z_1,u_1)\\ \cdots\\ z_{k+1}-z_k-\tau\phi(k\tau,z_k,u_k)\\ \cdots\\ z_T-z_{N-1}-\tau\phi((N-1)\tau,z_{N-1},u_{N-1})\end{array}\right).$$

Controllability of the linearization of $\dot{z}=\phi(t,z,u)$ implies that the Newton method is well-defined, whenever τ is sufficiently small!

Relative motion of two satellites (unilateral control oriented along the local geomagnetic field)



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