Primal-Dual subgradient method for Truss Topology Design

– Formulation of TTD

– Primal-Dual method for huge LP problem

– Primal-Dual method for Conic Optimization

Truss Topology Design:

Goal: *To design a truss of a given total weight best able to withstand the given load*

Assumptions:

- Nodal displacements are small and we neglect the second order terms;
- Local topology.

Consider a particular bar AB:

$$\text{elongation } dl = (dB - dA)^T (B - A)/\|B - A\|,$$

$$\text{tension } \tau = \kappa \frac{dl \times S_{AB}}{\|B - A\|} = \kappa \frac{dl \times t_{AB}}{\|B - A\|^2}$$

$$\text{energy} = \frac{tension \times elongation}{2} = \frac{\tau dl}{2} =$$

$$= \frac{1}{2} t_{AB} [(dB - dA)^T \beta_{AB}]^2,$$

where $\beta_{AB} = \sqrt{\kappa}(B - A)\|B - A\|^{-2}$.

Let
m be the number of nodes, n - the number of bars,
$t = (t_1, .., t_n)$ - volume vector,
$f \in R^m$ - load,
$v \in R^m$ - displacement.

Denote

$$b_i[\nu] = \begin{cases} \beta_{A_iB_i}, & \nu = \nu''(i), \\ -\beta_{A_iB_i}, & \nu = \nu'(i), \\ 0, & \text{otherwise} \end{cases}$$

For every node ν the component of the reaction forces by the ith bar is

$$-t_i(b_i^T v) b_i[\nu]$$

and the *collection of the reaction forces* at the nodes is

$$-\sum_{i=1}^{n} t_i(b_i^T v) b_i = -[\sum_{i=1}^{n} t_i b_i b_i^T] v = -A(t)v,$$

where $A(t) = \Sigma_{i=1}^{n} t_i b_i b_i^T$ - *bar-stiffness matrix* of the truss.

At equilibrium the reaction forces must compensate the external ones:

$$A(t)v = f.$$

Compliance - the potential energy stored by the truss at equilibrium is

$$\frac{1}{2}\sum_{i=1}^{n} t_i[v[\nu''(i)] - v[\nu'(i)])^T \beta_{A_iB_i}]^2 = \frac{1}{2}\sum_{i=1}^{n} t_i(v^T b_i)^2 =$$

$$= \frac{1}{2}v^T[\sum_{i=1}^{n} t_i b_i b_i^T]v = \frac{1}{2}v^T A(t)v = \frac{1}{2}f^T v.$$

Truss Topology Design Problem
Given a local structure

$$m, n; \{b_i \in R^m\}_{i=1}^{n}$$

a load $f \in R^m$*, and a total bar volume* $w > 0$*, find a truss* $t = (t_1, .., t_n)$*:*

$$\sum_{i=1}^{n} t_i = w$$

with the minimum possible compliance with respect to the load f.

Formulation of TTD as LP problem

Let us consider a primal problem:

$$\min_{u,w}\{< f, u >: A(w)v = f, w \geq 0, < e, w >= T\}, \quad (1)$$

and a "semi-primal" problem:

$$\max_{y}\{< f, y >: max_{1\leq i\leq N}| < a_i, y > | \leq 1\}. \quad (2)$$

Denote by u^*, w^* the optimal solution of (1)and by y^* the optimal solution of (2).

Then there exist multipliers $x^* \in R_+^N$:

$$f = \sum_{i\in J_+} a_i x_i^* - \sum_{i\in J_-} a_i x_i^*, \quad x_i^* = 0, \ \notin J_+ \cap J_-, \quad (3)$$

where $J_+ = \{i :< a_i, y^* >= 1\}$, $J_- = \{i :< a_i, y^* >= -1\}$.

Lemma 1 *Let*

$$w^* = \frac{T}{< e, x^* >} x^*, \qquad u^* = \frac{< e, x^* >}{T} y^*. \quad (4)$$

Then the pair (w^*, u^*), defined by (4), is feasible and optimal solution for (1).

Primal-dual subgradient method for LP

Consider the following primal-dual pair of Linear Programming problems:

$$f^* = \min_{x \in R^n} \{< c, x >: \; Ax = b, \; x \geq 0\} = \qquad (5)$$

$$= \max_{s \in R^n, y \in R^m} \{< b, y >: s = c - A^T y \geq 0\},$$

where $c \in R^n, \; b \in R^m, \; A \in R^{m \times n}$.

Assume that both problems in (5) are solvable.
Thus, there exist $x^* \geq 0, \; y^* \in R^m$:

$$Ax^* = b, \quad s^* = c - A^T y^*, \quad < s^*, x^* >= 0. \qquad (6)$$

For $y =\in R^m$, denote

$$j(y) : \quad \frac{< Ae_{j(y)}, y > -c^{j(y)}}{\|Ae_{j(y)}\|} = g(y) =$$

$$\max_{1 \leq j \leq n} \frac{< Ae_j, y > -c^j}{\|Ae_j\|},$$

$$g'(y) = \frac{Ae_{j(y)}}{\|Ae_{j(y)}\|}, \quad \|g'(y)\| = 1.$$

The method generates the main minimization sequence in the dual space, constructing at the same time an approximation primal solution. In both cases, the linear equality and inequality constraints can be violated.

Scheme of SG(h):

Let $y_0 = 0 \in R^m$ and step $h > 0$.

For $k > 0$ *do:*

if $g(y_k) \leq h$*, then (F):* $y_{k+1} = y_k + h\frac{b}{\|b\|}$,

$$\textit{else (G)}: \quad y_{k+1} = y_k - g(y_k)g'(y_k). \tag{7}$$

Let us define the approximations for the primal and dual solutions as follows:

$$\bar{x}_N = \frac{\|b\|}{hN_f} \sum_{k \in \mathcal{G}_N} \frac{g(y_k)}{\|Ae_{j(y_k)}\|} e_{j(y_k)},$$

$$\bar{y}_N = \frac{1}{N_f} \sum_{k \in \mathcal{F}_N} y_k, \quad \bar{s}_N = c - A^T \bar{y}_N. \tag{8}$$

Motivation:

Denote by $d_A : d_A^j = \|Ae_j\|, \; j = 1, .., n.$
In view of (F), we have

$$A^T y_k \le c + h d_A, \quad k \in \mathcal{F}_N.$$

Then,

$$\bar{s}_N = c - \frac{1}{N_f} \sum_{k \in \mathcal{F}_N} A^T y_k \ge -h d_A,$$

$$y_{N+1} = y_0 + \frac{h N_f}{\|b\|} b - \sum_{k \in \mathcal{G}_N} \frac{g(y_k)}{\|Ae_{j(y_k)}\|} Ae_{j(y_k)} =$$

$$= \frac{h N_f}{\|b\|}(b - A\bar{x}_N).$$

Theorem 1 *Denote* $D = 2\left(\frac{<x^*, d_A>}{\|b\|} + 1\right)$. *For any* $N \geq 0$ *we have:*

$$N_f \geq \frac{1}{D}\left(N + 1 - \frac{\|y^*\|^2}{h^2}\right).$$

If $N_f \geq 1$, *then*

$$< c, \bar{x}_N > - < b, \bar{y}_N > \leq \frac{1}{2} h \|b\|.$$

Finally, if $N + 1 > \frac{\|y^*\|^2}{h^2}$,

$$< x^*, \bar{s}_N > + < \bar{x}_N, s^* > \leq h\|b\|,$$

and the residual in the primal-dual system vanishes as $N \to \infty$:

$$\frac{1}{\|b\|}\|b - A\bar{x}_N\| \leq \sqrt{\frac{D}{N_f}} + \frac{\|y^*\|}{hN_f}.$$

Implementable version of the method

Given accuracy parameters $\epsilon_f, \epsilon_g, \epsilon_a$.
Goal: generate an approximate primal-dual solution $(\hat{x}, \hat{y}, \hat{s})$:

$$\hat{x} \geq 0, \quad \hat{s} = c - A^T \hat{y} \geq -\epsilon_g,$$

$$< c, \hat{x} > - < b, \hat{y} > \leq \epsilon_f, \quad \|A\hat{x} - b\| \leq \epsilon_a.$$

Let

$$h = \min \left\{ \epsilon_f \frac{2}{\|b\|}, \epsilon_g \frac{1}{\max_{1 \leq j \leq n} \|Ae_j\|} \right\}.$$

Stopping criterion:

$$\|A\bar{x}_N - b\| \leq \epsilon_a.$$

Solving Huge-Scale LP by the Method

Assumption: the data of the problem is uniformly sparse:

$$p(c) \leq r, \quad p(A^T e_i) \leq r, \ i = 1, .., m,$$

$$p(b) \leq q, \quad p(Ae_j) \leq q, \ j = 1, .., n,$$

$$r << n, \quad q << m.$$

Initial step:

- Compute the norm $\|b\|$ $\quad O(q)$ a.o.;
- Compute norms $\|Ae_j\|, \ j = 1, .., n$ $\quad \mathbf{O(p(A))}$ a.o.;
- Set $y_0 = 0, \ \hat{x}_0 = 0, \ u_0 = A^T y_0 - c$ $\quad (m + 2n)$ a.o.;
- Fill the binary table for computing the value $g(y)$ by vector u_0 $\quad$ (n operations).
- Choose the step h $\quad O(n)$ a.o.

k-th step: At the beginning the values $g(y_k), u_k$ *are already computed.*

- Update of y_k by a sparse direction u_k: $y_{k+1} = y_k + u_k$;

$$u_k = h\frac{b}{\|b\|} \text{ or } u_k = -\frac{g(y_k)}{\|Ae_{j(y_k)}\|}Ae_{j(y_k)} \longrightarrow$$

$$\longrightarrow p(u_k) \leq q \longrightarrow O(q)\text{a.o.};$$

- Compute the new residual u_{k+1} in parallel with updating $g(y_{k+1})$. We start with $u_+ = u_k$

  For $j \in \sigma(u_k),\ i \in \sigma(A^T e_i)$ *iterate:*
  1. Update $u_+^i = u_+^i + A_{i,j}u_k^j$;
  2. Update the value $\max_{1\leq l\leq n} u_+^l$ by the binary table $\longrightarrow O(rqlog_2 n)$

  In the end, we set $y_{k+1} = u_+$
  and $g(y_{k+1}) = \max_{1\leq l\leq n} u_+^l$.

- Update $\|y_{k+1}\|^2$ for the stopping criterion

$$\|y_{k+1}\|^2 \leq \left(\epsilon_a \frac{hk_f}{\|b\|}\right)^2.$$

  Since $\|y_{k+1}\|^2 = \|y_k\|^2 + 2 < y_k, u_k > + \|u_k\|^2 \longrightarrow O(q)\text{a.o.}$

Primal-dual subgradient method for Linear Conic LP

Assume:

- Space of primal variables E is partitioned as follows:

$$x^j \in E_j, \ j = 1, .., n \ (x^1, .., x^n) \in E,$$

where E_j - are some finite-dimensional spaces. Thus,

$$dim E = \sum_{j=1}^{n} dim E_j$$

$$< c, x >= \sum_{j=1}^{n} < c^j, x^j > \ \text{for } c \in E^*$$

- For linear operator $A : E \to R^m$

$$A = (A_1, .., A_n), Ax = \sum_{j=1}^{n} A_j x^j, \ x^j \in E.$$

- For conic constraint $x \in K$,

$$K = \bigotimes_{j=1}^{n} K_j,$$

where all $K_j \subset E_j$ are closed convex pointed cones. Thus,

$$K^* = \bigotimes_{j=1}^{n} K_j^*.$$

Primal Problem of Linear Conic Optimization

$$f_* = inf_{x \in K} < c, x >: \ Ax = b. \tag{9}$$

Dual Problem of Linear Conic Optimization

$$sup_{y \in R^m, \ s \in K^*} < b, y >: s + A^* y = c. \tag{10}$$

Assumptions:

- $$c \in intK^*$$
- The dual problem (10)is solvable.

Denote by y^* one of its solution, and $s_* = c - A^T y_* \in K^*$ the corresponding slack variables. Thus,

Duality Theorem: *the primal problem (9) is solvable and for primal-dual pair (9)-(10) there is no duality gap:* $< s_*, x_* >= 0$.

Functional Form of the Dual Constraints

Constraints of the dual problem in a separable form:

$$sup_{y\in R^m,\ s\in E^*}\left\{< b, y >:\ \ s^j = c^j - A_j^* y \in K_j^*,\ j = 1, .., n\right\}. \tag{11}$$

Consider the following function

$$\psi_j(u^j) = \min_{\tau} \tau : \tau c^j - u^j \in K_j^*. \tag{12}$$

Lemma 2 *Function $\psi_j(u^j)$ is convex on E_j. It has the following representation:*

$$\psi_j(u^j) = \max_{x^j\in E_j}\{< u^j, x^j >:\ \ < c^j, x^j >= 1,\ x^j \in K_j\}. \tag{13}$$

Thus,

$$\partial\psi_j(u^j) = Argmax_{x^j\in E_j}\{< u^j, x^j >:\ \ < c^j, x^j >= 1,\ x^j \in K_j$$

It is clear that $c^j - A_j^* y \in K_j^*$ iff $f_j(y) \equiv \psi_j(A_j^* y) \le 1$.

Normalizing subgradients of function f_j

For each u^j, denote by $x^j(u^j) \in K_j$ an arbitrary oprimal solution to (13).

Then,

$$f_j'(y) = A_j x^j(A_j^* y) \in \partial f_j(y) \subset R^m \quad (14)$$

$$\|f_{j(y)}'\| \leq \max_{x^j \in K_j} \{\|A_j x^j\| : \ < c^j, x^j >= 1\} = M_j. \quad (15)$$

Denote

$$M_* = \max_{1 \leq j \leq n} M_j \quad \text{and } g_j(y) = \frac{1}{\|f_j'(y)\|}(f_j(y) - 1).$$

Dual problem: Representation in functional form:

$$sup_{y\in R^m, s\in E^*} \left\{ < b, y >: \; g(y) = max_{1\le j\le n} g_j(y) \le 0 \right\}. \tag{16}$$

Denote $j(y) : g_j(y) = g(y)$.
Then,

$$g'(y) = \frac{A_{j(y)} x^{j(y)} (A^*_{j(y)} y)}{\|A_{j(y)} x^{j(y)} (A^*_{j(y)} y)\|}, \quad \|g'(y)\| = 1.$$

Primal-Dual Subgradient Method

The method generates the main minimization sequence in the dual space, constructing at the same time an approximation primal solution. In both cases, the linear equality and inequality constraints can be violated.

Scheme of SG(h):

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For $k > 0$ *do*:

if $g(y_k) \leq h$, then (F): $y_{k+1} = y_k + h\frac{b}{\|b\|}$,

$$\text{else (G)}: \; y_{k+1} = y_k - g(y_k)g'(y_k). \tag{17}$$

Let us define approximations for the optimal primal-dual solutions as follows:

$$\bar{x}_N = \frac{\|b\|}{hN_f} \sum_{k \in \mathcal{G}_N} \frac{g(y_k) I_{j(y_k)}(x^{j(y_k)}(A^*_{j(y_k)} y_k))}{\|A_{j(y_k)} x^{j(y_k)}(A^*_{j(y_k)} y_k)\|} \in K,$$

$$\bar{y}_N = \frac{1}{N_f} \sum_{k \in \mathcal{F}_N} y_k, \quad \bar{s}_N = c - A^T \bar{y}_N,$$

where vector $I_j(x^j)$:

$$I^i_j(x^j) = \begin{cases} x^j, \; if i = j, \\ 0, \; otherwise, \end{cases} \; i = 1, .., n.$$

Motivation:

$$\bar{s}_N^j = c^j - \frac{1}{N_f} \sum_{k \in \mathcal{F}_N} A_j^* y_k \succeq_{K_j^*} -hM_j c^j,$$

$$\bar{y}_{N+1} = \frac{hN_f}{\|b\|} b - \sum_{k \in \mathcal{G}_N} \frac{g(y_k) \cdot A \cdot I_{j(y_k)}(x^{j(y_k)}(A_{j(y_k)}^* y_k))}{\|A_{j(y_k)} x^{j(y_k)}(A_{j(y_k)}^* y_k)\|}$$

Theorem 2 *Denote* $\hat{D} = 2\left(\frac{<\hat{c}, x^*>}{\|b\|} + 1\right) \leq 2\left(\frac{M_*}{\|b\|} < \hat{c}, x^* > +1$ *where vector* $\hat{c} \in K^* : \ \hat{c}^j = M_j c^j, \ j = 1, .., n.$

For any $N \geq 0$ *we have:*

$$N_f \geq \frac{1}{\hat{D}} \left(N + 1 - \frac{\|y^*\|^2}{h^2} \right).$$

If $N_f \geq 1$, *then*

$$< c, \bar{x}_N > - < b, \bar{y}_N > \leq \frac{1}{2} h \|b\|.$$

Finally, if $N + 1 > \frac{\|y^*\|^2}{h^2}$,

$$< x^*, \bar{s}_N > + < \bar{x}_N, s^* > \leq h \|b\|,$$

and the residual in the primal-dual system vanishes as $N \to \infty$*:*

$$\frac{1}{\|b\|} \|b - A\bar{x}_N\| \leq \sqrt{\frac{\hat{D}}{N_f}} + \frac{\|y^*\|}{hN_f}.$$