Lecture 8.

Applications of Structural Programming

– Bounds on the parameter of a s.-c. barrier.

– Linear and Quadratic Programming.

– Semidefinite Programming.

– Extremal Ellipsoids.

– Separable Problems.

– Geometric Programming.

– Approximation in L_p norms.

– Choice the an Optimization Scheme.

Problem: (Mediator)

$$\begin{aligned} & \min \langle c, x \rangle, \\ \text{s.t.} \;\; & x \in Q. \end{aligned} \tag{8.1}$$

We assume that:

- Q is a closed convex set with nonempty interior.
- We know a ν-self-concordant barrier $F(x)$ for Q.

Complexity estimate:

$$O\left(\sqrt{\nu} \cdot \ln \frac{1}{\epsilon}\right)$$

iterations of a path-following scheme.

Complexity of each iteration:

Solve a system of linear equations.

Lower bounds for ν

Lemma 8.1 *Let $f(t)$ be a ν-self-concordant barrier for $(\alpha,\beta) \subset R^1$, $\alpha < \beta < \infty$. Then*

$$\nu \geq \kappa \equiv \sup_{t\in(0,T)} \frac{(f'(t))^2}{f''(t)} \geq 1.$$

Proof. Note that $\nu \geq \kappa$ by definition. Let us assume that $\kappa < 1$.

Let $\bar{\alpha} \in (\alpha,\beta)$ and $f'(\bar{\alpha}) > 0$. Consider the function

$$\phi(t) = \frac{(f'(t))^2}{f''(t)}, \quad t \in [\bar{\alpha},\beta).$$

Then

$$\phi'(t) = 2f'(t) - \left(\frac{f'(t)}{f''(t)}\right)^2 f'''(t)$$

$$= f'(t)\left(2 - \frac{f'(t)}{\sqrt{f''(t)}} \cdot \frac{f'''(t)}{[f''(t)]^{3/2}}\right) \geq 2(1-\sqrt{\kappa})f'(t).$$

Therefore, for all $t \in [\bar{\alpha},\beta)$ we have:

$$\kappa \geq \phi(t) \geq \phi(\bar{\alpha}) + 2(1-\sqrt{\kappa})(f(t) - f(\bar{\alpha})).$$

That is a contradiction. □

Corollary 8.1 *Let $F(x)$ be a ν-self-concordant barrier for $Q \subset R^n$. Then $\nu \geq 1$.*

Proof. Consider $f(t) = F(x+tu)$, $x \in \operatorname{int} Q$, $u \neq 0$. □

Let Q be a closed convex set with nonempty interior.

Consider $\bar{x} \in \operatorname{int} Q$. Assume that there is a set of *recession* directions $\{p_1, \dots, p_k\}$:

$$\bar{x} + \alpha p_i \in Q \quad \forall \alpha \geq 0.$$

Theorem 8.1 *Let the positive coefficients* $\{\beta_i\}_{i=1}^k$ *satisfy the condition*

$$\bar{x} - \beta_i \, p_i \notin \operatorname{int} Q, \quad i = 1, \dots, k.$$

If for some positive $\alpha_1, \dots, \alpha_k$ *we have*

$$\bar{y} = \bar{x} - \sum_{i=1}^{k} \alpha_i p_i \in Q,$$

then the parameter ν *of any self-concordant barrier for* Q *satisfies the inequality:*

$$\nu \geq \sum_{i=1}^{k} \frac{\alpha_i}{\beta_i}.$$

Proof:
Let $F(x)$ be a ν-s.c.b. for Q. Since p_i is a recession direction, we have:

$$\langle F'(\bar{x}), -p_i \rangle \geq \langle F''(\bar{x}) p_i, p_i \rangle^{1/2} \equiv \| p_i \|_{\bar{x}},$$

(Theorem 6.9). Since $\bar{x} - \beta_i \, p_i \notin Q$, we have also

$$\beta_i \| p_i \|_{\bar{x}} \geq 1,$$

(Theorem 6.5). Therefore, in view of Theorem 7.4,

$$\nu \geq \langle F'(\bar{x}), \bar{y} - \bar{x} \rangle = \langle F'(\bar{x}), -\sum_{i=1}^{k} \alpha_i p_i \rangle$$

$$\geq \sum_{i=1}^{k} \alpha_i \| p_i \|_{\bar{x}} \geq \sum_{i=1}^{k} \frac{\alpha_i}{\beta_i}. \qquad \square$$

Upper bound for ν

Let Q be a closed convex set with nonempty interior. Assume that Q contains no straight line.

Let $\bar{x} \in \operatorname{int} Q$. Define the *polar set*

$$P(\bar{x}) = \{s \in R^n \mid \langle s, x - \bar{x} \rangle \leq 1, \quad \forall x \in Q\}.$$

Denote $V(x) = \operatorname{vol}_n P(x)$.

Theorem 8.2 *There exist absolute constants c_1 and c_2, such that the function*

$$F(x) = c_1 \cdot \ln V(x)$$

is a $(c_2 \cdot n)$-self-concordant barrier for Q.

(Accept without proof.)

$F(x)$ is called the *universal barrier* for Q

Note:

- In general, $F(x)$ cannot be computed easily.
- The analytical complexity of the problem (8.1), equipped by the universal barrier, is

  $$O\left(\sqrt{n} \cdot \ln \frac{1}{\epsilon}\right).$$

  Such efficiency is *impossible*, if we use local black-box oracle (Theorem 6.5).

Linear Programming

Problem formulation:

$$\begin{aligned} &\min \langle c, x \rangle \\ \text{s.t }\ & Ax = b, \\ & x^{(i)} \geq 0,\ i = 1 \ldots n, \quad (\Leftrightarrow x \in R^n_+) \end{aligned} \tag{8.2}$$

where A is an $(m \times n)$-matrix, $m < n$.

Barrier:

$$F(x) = - \sum_{i=1}^{n} \ln x^{(i)},$$

$$\nu = n,$$

(see Example 7.3 and Theorem 7.2).

Note:

- We use the restriction of $F(x)$ on $\{x : \ Ax = b\}$.
- This restriction is an n-self-concordant barrier (Theorem 7.3).
- The complexity estimate for the problem (8.2) is

$$O\left(\sqrt{n} \cdot \ln \frac{1}{\epsilon}\right)$$

iterations of the path-following scheme.

Lemma 8.2 *The parameter ν of any self-concordant barrier for R^n_+ satisfies the inequality*

$$\nu \geq n.$$

Proof:
Let us choose

$$\begin{aligned} \bar{x} &= e \equiv (1, \dots, 1) \in \operatorname{int} R^n_+, \\ p_i &= e_i, \quad i = 1, \dots, n, \end{aligned}$$

where e_i is the ith coordinate ort of R^n.

Clearly, the conditions of Theorem 8.1 are satisfied with

$$\alpha_i = \beta_i = 1, \quad i = 1, \dots, n.$$

Therefore

$$\nu \geq \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} = n.$$

□

Note:

- The value of the parameter of the restriction of $F(x)$ on $\{x : \ Ax = b\}$ can be better.

Quadratic Programming

Problem formulation:

$$
\begin{aligned}
&\min\ q_0(x) = \alpha_0 + \langle a_0, x\rangle + \tfrac{1}{2}\langle A_0 x, x\rangle \\
\text{s.t}\ & q_i(x) = \alpha_i + \langle a_i, x\rangle + \tfrac{1}{2}\langle A_i x, x\rangle \leq \beta_i, \\
& i = 1 \ldots m, \\
& x \in R^n,
\end{aligned}
\tag{8.3}
$$

where A_i are some positive semidefinite $(n \times n)$-matrices.

Mediator:

$$
\begin{aligned}
&\min_{x,t}\ t_0 \\
\text{s.t}\ & q_i(x) \leq t_i,\ i = 0 \ldots m, \\
& t_i \leq \beta_i,\ i = 1 \ldots m, \\
& x \in R^n,\ t \in R^{m+1}.
\end{aligned}
\tag{8.4}
$$

Barrier:

$$F(x,t) = -\sum_{i=0}^{m} \ln(t_i - q_i(x)) - \sum_{i=1}^{m} \ln(\beta_i - t_i),$$

$$\nu = 2m + 1,$$

(see Examples 7.3, 7.4, and Theorem 7.2).

Note:

- The complexity estimate for the problem (8.3) is $O\left(\sqrt{2m+1}\right.$ iterations of the path-following scheme.
- The number of iterations *does not depend* on n.

Nonsmooth quadratic components

We can treat also the components $\| A_i x - b_i \|$.

Lemma 8.3 *Function* $F(x,t) = -\ln(t^2 - \| x \|^2)$ *is a 2-self-concordant barrier for the convex set*

$$K_2 = \{(x,t) \in R^{n+1} \mid \; t \geq \| x \|\}.$$

Proof. Let $z = (x,t) \in \operatorname{int} K_2$, $u = (h,\tau) \in R^{n+1}$. Consider $\phi(\alpha) = F(z + \alpha u)$. Denote $\phi^{(\cdot)} = \phi^{(\cdot)}(0)$,

$$\omega(\alpha) = (t + \alpha\tau)^2 - \| x + \alpha h \|^2, \quad \omega^{(\cdot)} = \omega^{(\cdot)}(0),$$

$$\omega'(0) = 2(t\tau - \langle x, h \rangle), \quad \omega''(0) = 2(\tau^2 - \| h \|^2).$$

Then $\phi' = -\frac{\omega'}{\omega}$, $\phi'' = \left(\frac{\omega'}{\omega}\right)^2 - \frac{\omega''}{\omega}$, $\phi''' = 3\frac{\omega'\omega''}{\omega^2} - 2\left(\frac{\omega'}{\omega}\right)^3$.

Note: $2\phi'' \geq (\phi')^2 \;\Leftrightarrow\; (\omega')^2 \geq 2\omega\omega''$. That is

$$(t\tau - \langle x, h \rangle)^2 - (t^2 - \| x \|^2)(\tau^2 - \| h \|^2) \geq 0. \quad (8.5)$$

We can restrict ourselves by $\tau > \| h \|$. Then the worst h: $\langle x, h \rangle = \| x \| \| h \|$ and (8.5) holds.

Further,

$$\frac{| \phi''' |}{(\phi'')^{3/2}} = 2\frac{| \omega' | \cdot | (\omega')^2 - \frac{3}{2}\omega\omega'' |}{[(\omega')^2 - \omega\omega'']^{3/2}} \leq 2$$

since $0 \leq \frac{\omega\omega''}{(\omega')^2} \leq \frac{1}{2}$ and $[1 - \xi]^{3/2} \geq 1 - \frac{3}{2}\xi$. □

Lemma 8.4 *The parameter ν of any self-concordant barrier for the set K_2 satisfies the inequality*

$$\nu \geq 2.$$

Proof. Let us choose

$$\bar{z} = (0,1) \in \operatorname{int} K_2,$$

$$\alpha_1 = \alpha_2 = \tfrac{1}{2},$$

$$\beta_1 = \beta_2 = \tfrac{1}{2},$$

$$p_1 = (h,1), \quad p_2 = (-h,1),$$

where h is an arbitrary vector in R^n, $\| h \| = 1$.

Note that

$$\bar{z} + \gamma p_i = (\pm\gamma h, 1+\gamma) \in K_2, \quad \forall \gamma \geq 0,$$

$$\bar{z} - \beta_i p_i = (\pm\tfrac{1}{2}h, \tfrac{1}{2}) \in \partial K,$$

$$\bar{z} - \alpha_1 p_1 - \alpha_2 p_2 = (-\tfrac{1}{2}h + \tfrac{1}{2}h, 1 - \tfrac{1}{2} - \tfrac{1}{2}) = 0 \in K_2.$$

Therefore, the conditions of Theorem 8.1 are satisfied and

$$\nu \geq \tfrac{\alpha_1}{\beta_1} + \tfrac{\alpha_2}{\beta_2} = 2.$$

□

Semidefinite Programming

Let $X = \{x^{(i,j)}\}_{i=1,n}^{j=1,n}$ be a symmetric $n \times n$-matrix ($X \in S^{n\times n}$).

Scalar Product: $X, Y \in S^{n\times n}$,

$$\langle X, Y\rangle_F = \sum_{i=1}^{n} \sum_{j=1}^{n} x^{(i,j)} y^{(i,j)}, \quad \| X \|_F = \langle X, X\rangle_F^{1/2}.$$

Note: $\forall X, Y \in S^{n\times n}$ we have:

$$\langle X, Y \cdot Y\rangle_F = \langle YXY, I_n\rangle_F = \mathrm{Trace}\,(YXY).$$

Positive semidefinite matrices: $X \in P_n$ iff

$$\forall u \in R^n \quad \langle Xu, u\rangle \geq 0.$$

P_n is a convex closed set. $X \in \mathrm{int}\, P_n$ iff

$$\forall u \in R^n, u \neq 0, \quad \langle Xu, u\rangle > 0.$$

Problem formulation:

$$\begin{aligned} &\min \langle C, X\rangle_F \\ \text{s.t } &\langle A_i, X\rangle_F = b_i, \ i = 1 \ldots m, \\ &X \in P_n, \end{aligned} \tag{8.6}$$

where C and A_i belong to $S^{n\times n}$.

What could be a self-concordant barrier for P_n?

Let $X \in \operatorname{int} K_n$. Denote $F(X) = -\ln\det X$.

Lemma 8.5 *1. $F(X)$ is convex, $F'(X) = -X^{-1}$.*

2. For any $\Delta \in S^{n\times n}$ we have:

$$\langle F''(X)\Delta, \Delta\rangle_F = \| X^{-1/2}\Delta X^{-1/2} \|_F,$$

$$D^3F(x)[\Delta,\Delta,\Delta] = -2\langle I_n, [X^{-1/2}\Delta X^{-1/2}]^3\rangle_F.$$

Proof. 1. Let $\Delta \in S^{n\times n}$ and $X, X+\Delta \in P_n$. Then

$$F(X+\Delta) - F(X) = -\ln\det(X+\Delta) - \ln\det X$$

$$= -\ln\det(I_n + X^{-1/2}\Delta X^{-1/2})$$

$$\geq -\ln\left(\tfrac{1}{n}\operatorname{Trace}\left(I_n + X^{-1/2}\Delta X^{-1/2}\right)\right)^n$$

$$= -n\ln\left(1 + \tfrac{1}{n}\langle I_n, X^{-1/2}\Delta X^{-1/2}\rangle_F\right)$$

$$\geq -\langle I_n, X^{-1/2}\Delta X^{-1/2}\rangle_F = -\langle X^{-1}, \Delta\rangle_F.$$

Thus, F is convex and $F'(x) = -X^{-1}$.

2. Consider $\phi(\alpha) = \langle F'(X+\alpha\Delta), \Delta\rangle_F$. Then

$$\phi(\alpha) - \phi(0) = \langle X^{-1} - (X+\alpha\Delta)^{-1}, \Delta\rangle_F$$

$$= \langle (X+\alpha\Delta)^{-1}[(X+\alpha\Delta) - X]X^{-1}, \Delta\rangle_F$$

$$= \alpha\langle (X+\alpha\Delta)^{-1}\Delta X^{-1}, \Delta\rangle_F.$$

Thus, $\phi'(0) = \langle F''(X)\Delta, \Delta\rangle_F = \langle X^{-1}\Delta X^{-1}, \Delta\rangle_F$.

For the last expression consider

$\psi(\alpha) = \langle (X+\alpha\Delta)^{-1}\Delta(X+\alpha\Delta)^{-1}, \Delta\rangle_F.$ $\square$

Theorem 8.3 $F(X)$ *is an n-self-concordant barrier for* P_n.

Proof. Let $X \in \operatorname{int} P_n$ and $\Delta \in S^{n \times n}$. Denote

$$Q = X^{-1/2} \Delta X^{-1/2}, \quad \lambda_i = \lambda_i(Q),\ i = 1, \ldots, n.$$

Then

$$-\langle F'(X), \Delta \rangle_F = \langle X^{-1}, \Delta \rangle_F = \operatorname{Trace}(Q) = \sum_{i=1}^{n} \lambda_i,$$

$$\langle F''(X)\Delta, \Delta \rangle_F = \langle I_n, Q^2 \rangle_F = \operatorname{Trace}(Q^2) = \sum_{i=1}^{n} (\lambda_i)^2,$$

$$D^3 F(X)[\Delta, \Delta, \Delta] = -2 \langle I_n, Q^3 \rangle_F = -2 \sum_{i=1}^{n} (\lambda_i)^3.$$

Using two inequalities

$$\left(\sum_{i=1}^{n} \lambda_i\right)^2 \leq n \sum_{i=1}^{n} \lambda_i^2,$$

$$\left| \sum_{i=1}^{n} \lambda_i^3 \right| \leq \left[\sum_{i=1}^{n} \lambda_i^2\right]^{3/2},$$

we get

$$\langle F'(X), \Delta \rangle_F^2 \leq n \langle F''(X)\Delta, \Delta \rangle_F,$$

$$| D^3 F(X)[\Delta, \Delta, \Delta] | \leq 2 \langle F''(X)\Delta, \Delta \rangle_F^{3/2}.$$

□

Lemma 8.6 *The parameter ν of any self-concordant barrier for P_n satisfies the inequality*

$$\nu \geq n.$$

Proof:
Let us choose

$$\begin{aligned} \bar{X} &= I_n \in \operatorname{int} P_n, \\ p_i &= e_i e_i^T, \quad i = 1, \ldots, n, \end{aligned}$$

where e_i is the ith coordinate ort of R^n.

Note that the conditions of Theorem 8.1 are satisfied with

$$\alpha_i = \beta_i = 1, \quad i = 1, \ldots, n.$$

Indeed,

$$I_n - e_i e_i^T \in \partial P_n, \quad I_n - \sum_{i=1}^{n} e_i e_i^T = 0 \in P_n.$$

Therefore

$$\nu \geq \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i} = n.$$

$\square$

Remarks

- We use the restriction of $F(X)$ onto the set
$$\{X : \langle A_i, X\rangle_F = b_i,\ i = 1\ldots m\}.$$
- This restriction is an n-self-concordant barrier (Theorem 7.3).
- The complexity estimate for the problem (8.6) is
$$O\left(\sqrt{n}\cdot \ln\frac{1}{\epsilon}\right)$$
iterations of the path-following scheme.

  The dimension of the problem (8.6) is $\frac{1}{2}n(n+1)$.
- Using the barrier $-\ln\det X$ we can treat nonsmooth components:
$$\max_{1\le i\le n} \lambda_i(\mathcal{A}(x)) \le t,$$
where the matrix $\mathcal{A}(x) \in S^{n\times n}$ depends linearly on x.

  For that we can use the barrier
$$F(x,t) = -\ln\det(tI_n - \mathcal{A}(x)),$$
which is self-concordant with $\nu = n$.

Extremal Ellipsoids

1. Circumscribed ellipsoid.

Given by a set of points $a_1, \ldots, a_m \in R^n$, find an ellipsoid W such that

$$a_i \in W, \quad i = 1, \ldots, m,$$

and which volume is as small as possible.

Let $H \in \text{int}\, P_n$, $v \in R^n$. We can represent W as follows:

$$W = \{x \in R^n \mid x = H^{-1}(v + u), \ \| u \| \leq 1\}.$$

Then

$$a \in W \quad \Leftrightarrow \quad \| Ha - v \| \leq 1.$$

Note that

$$\text{vol}_n W = \text{vol}_n B_2(0, 1) \det H^{-1} = \frac{\text{vol}_n B_2(0, 1)}{\det H}.$$

Thus, our problem is

$$\begin{aligned} & \min_{H, v, \tau} \ \tau, \\ \text{s.t.} \ & -\ln \det H \leq \tau, \\ & \| Ha_i - v \| \leq 1, \ i = 1, \ldots, m, \\ & H \in P_n, \ v \in R^n, \tau \in R^1. \end{aligned} \tag{8.7}$$

Lemma 8.7 *The function*

$$-\ln\det H - \ln(\tau + \ln\det H)$$

is an $(n+1)$*-self-concordant barrier for the set*

$$\{(H,\tau) \in S^{n\times n} \times R^1 \mid \ \tau \geq -\ln\det H,\ H \in P_n\}.$$

(Accept without proof.)

Barrier for (8.7):

$$\begin{aligned} F(H,v,\tau) = & -\ln\det H - \ln(\tau + \ln\det H) \\ & - \sum_{i=1}^{m} \ln(1 - \parallel Ha_i - v \parallel^2), \\ \nu = & \ m + n + 1. \end{aligned}$$

Efficiency estimate:

$$O\left(\sqrt{m+n+1} \cdot \ln\frac{1}{\epsilon}\right)$$

iterations of a path-following scheme.

2. Inscribed ellipsoid with fixed center.

Let Q be a convex polytope defined by a set of linear inequalities:

$$Q = \{x \in R^n \mid \langle a_i, x\rangle \le b_i,\ i = 1, \ldots, m\},$$

and let $v \in \operatorname{int} Q$.

Find an ellipsoid $W \subset Q$, centered at v, with the maximal volume.

Let $H \in \operatorname{int} P_n$. We can represent W as follows:

$$W = \{x \in R^n \mid \langle H^{-1}(x - v), x - v\rangle \le 1\}.$$

Lemma 8.8 *Let $\langle a, v\rangle < b$. The inequality*

$$\langle a, x\rangle \le b$$

is valid for any $x \in W$ if and only if

$$\langle Ha, a\rangle \le (b - \langle a, v\rangle)^2.$$

Proof. In Corollary 7.1 we have shown that

$$\max_u \{\langle a, u\rangle \mid \langle H^{-1}u, u\rangle \le 1\} = \langle Ha, a\rangle^{1/2}.$$

Therefore we need

$$\max_{x\in W} \langle a, x\rangle = \max_{x \in W}[\langle a, x - v\rangle + \langle a, v\rangle]$$

$$= \langle a, v\rangle + \max_x \{\langle a, u\rangle \mid \langle H^{-1}u, u\rangle \le 1\}$$

$$= \langle a, v\rangle + \langle Ha, a\rangle^{1/2} \le b,$$

This proves our statement since $\langle a, v\rangle < b$. □

Note that

$$\mathrm{vol}_n\, W = \mathrm{vol}_n\, B_2(0,1)[\det H^{-1}]^{1/2} = \frac{\mathrm{vol}_n\, B_2(0,1)}{[\det H]^{1/2}}.$$

Thus, our problem is

$$\begin{aligned} & \min_{H,\tau} \ \tau, \\ \text{s.t.} \ & -\ln\det H \le \tau, \\ & \langle Ha_i, a_i\rangle \le (b_i - \langle a_i, v\rangle)^2, \qquad (8.8) \\ & i = 1, \dots, m, \\ & H \in P_n, \ \tau \in R^1. \end{aligned}$$

Barrier for (8.8):

$$\begin{aligned} F(H,\tau) = & -\ln\det H - \ln(\tau + \ln\det H) \\ & - \sum_{i=1}^{m} \ln[(b_i - \langle a_i, v\rangle)^2 - \langle Ha_i, a_i\rangle], \\ \nu = & \ m + n + 1. \end{aligned}$$

Efficiency estimate:

$$O\left(\sqrt{m+n+1} \cdot \ln \frac{1}{\epsilon}\right)$$

iterations of a path-following scheme.

2. Inscribed ellipsoid with free center.

Let Q be a convex polytope defined by a set of linear inequalities:

$$Q = \{x \in R^n \mid \langle a_i, x\rangle \le b_i, \ i = 1, \dots, m\},$$

and let $\operatorname{int} Q \neq \emptyset$.

Find an ellipsoid $W \subset Q$, which has the maximal volume.

Let $G \in \operatorname{int} P_n$, $v \in \operatorname{int} Q$. We can represent W as follows:

$$\begin{aligned} W &= \{x \in R^n \mid \| G^{-1}(x - v) \| \le 1\} \\ &= \{x \in R^n \mid \langle G^{-2}(x - v), x - v\rangle \le 1\}. \end{aligned}$$

In view of Lemma 8.8, the inequality

$$\langle a, x\rangle \le b$$

is valid for any $x \in W$ if and only if

$$\| Ga \|^2 \equiv \langle G^2 a, a\rangle \le (b - \langle a, v\rangle)^2.$$

That gives a convex region for (G, v):

$$\| Ga \| \le b - \langle a, v\rangle.$$

Note that

$$\mathrm{vol}_n W = \mathrm{vol}_n B_2(0,1) \det G^{-1} = \frac{\mathrm{vol}_n B_2(0,1)}{\det G}.$$

Thus, our problem is

$$\begin{aligned} &\min_{G,v,\tau} \ \tau, \\ \text{s.t.} \ &-\ln\det G \le \tau, \\ &\| Ga_i \| \le b_i - \langle a_i, v\rangle, \ i = 1, \dots, m, \\ &G \in P_n, \ v \in R^n, \ \tau \in R^1. \end{aligned} \tag{8.9}$$

Barrier for (8.9):

$$\begin{aligned} F(G, v, \tau) = \ & -\ln\det G - \ln(\tau + \ln\det G) \\ & - \sum_{i=1}^{m} \ln[(b_i - \langle a_i, v\rangle)^2 - \| Ga_i \|^2], \\ \nu = \ & 2m + n + 1. \end{aligned}$$

Efficiency estimate:

$$O\left(\sqrt{2m+n+1} \cdot \ln \frac{1}{\epsilon}\right)$$

iterations of a path-following scheme.

Separable Programming

Problem formulation:

$$\min_{x \in R^n} \; q_0(x) = \sum_{j=1}^{m_0} \alpha_{0,j} f_{0,j}(\langle a_{0,j}, x\rangle + b_{0,j})$$

$$\text{s.t} \;\; q_i(x) = \sum_{j=1}^{m_i} \alpha_{i,j} f_{i,j}(\langle a_{i,j}, x\rangle + b_{i,j}) \leq \beta_i, \qquad (8.10)$$

$$i = 1 \ldots m,$$

where $\alpha_{i,j}$ are some positive coefficients, $a_{i,j} \in R^n$ and $f_{i,j}(t)$ are convex functions of one variable.

Mediator:

$$\min_{x,y,t,\tau} \; \tau_0$$

$$\text{s.t} \;\; y_{i,j} = \langle a_{i,j}, x\rangle + b_{i,j}, \; i = 0 \ldots m, \; j = 1 \ldots m_i,$$

$$f_{i,j}(y_{i,j}) \leq t_{i,j}, \; i = 0 \ldots m, \; j = 1 \ldots m_i,$$

$$\sum_{j=1}^{m_i} \alpha_{i,j} t_{i,j} \leq \tau_i, \; i = 0, \ldots, m,$$

$$\tau_i \leq \beta_i, \; i = 1, \ldots, m,$$

$$x \in R^n, \; \tau \in R^{m+1} \; y, t \in R^M, \qquad (8.11)$$

where $M = \sum_{i=0}^{m} m_i$.

Note: We can construct a self-concordant barrier for (8.11), if we can do that for the sets

$$t \geq f_{i,j}(x), \quad (x,t) \in R^2.$$

Barriers for two-dimensional sets

1. Logarithm and Exponent.

The barrier

$$F_1(x,t) = -\ln x - \ln(\ln x + t)$$

is a 2-self-concordant barrier for the set

$$Q_1 = \{(x,t) \in R^2 \mid \ x > 0,\ t \geq -\ln x\}$$

and the barrier

$$F_2(x,t) = -\ln t - \ln(\ln t - x)$$

is a 2-self-concordant barrier for the set

$$Q_2 = \{(x,t) \in R^2 \mid \ t \geq e^x\}.$$

2. Entropy function.

The barrier

$$F_3(x,t) = -\ln x - \ln(t - x\ln x)$$

is a 2-self-concordant barrier for the set

$$Q_3 = \{(x,t) \in R^2 \mid \ x \geq 0,\ t \geq x\ln x\}.$$

3. Increasing power functions.

The barrier

$$F_4(x,t) = -2\ln t - \ln(t^{2/p} - x^2)$$

is a 4-self-concordant barrier for the set

$$Q_4 = \{(x,t) \in R^2 \mid \ t \geq \mid x \mid^p\}, \quad p \geq 1,$$

and the barrier

$$F_5(x,t) = -\ln x - \ln(t^p - x)$$

is a 2-self-concordant barrier for the set

$$Q_5 = \{(x,t) \in R^2 \mid \ x \geq 0, \ t^p \geq x\}, \quad 0 < p \leq 1.$$

4. Decreasing power functions.

The barrier

$$F_6(x,t) = -\ln t - \ln(x - t^{-1/p})$$

is a 2-self-concordant barrier for the set

$$Q_6 = \left\{(x,t) \in R^2 \mid \ x > 0, \ t \geq \frac{1}{x^p}\right\}, \quad p \geq 1,$$

and the barrier

$$F_7(x,t) = -\ln x - \ln(t - x^{-p})$$

is a 2-self-concordant barrier for the set

$$Q_7 = \left\{(x,t) \in R^2 \mid \ x > 0, \ t \geq \frac{1}{x^p}\right\}, \quad 0 < p < 1.$$

Note:

- The barriers for the sets $Q_1 - Q_3$ and $Q_5 - Q_7$ are *optimal.*

Example of the proof:

Lemma 8.9 *The parameter ν of any self-concordant barrier for the set*

$$Q = \left\{ (x^{(1)}, x^{(2)}) \in R^2 \mid \ x^{(1)} > 0, \ x^{(2)} \geq \tfrac{1}{(x^{(1)})^p} \right\},$$

$p > 0$, satisfies the inequality $\nu \geq 2$.

Proof:
Let us fix some $\gamma > 1$ and set

$$\begin{aligned} \bar{x} &= (\gamma, \gamma) \in \operatorname{int} Q, \\ p_1 &= e_1, \quad p_2 = e_2, \\ \beta_1 &= \beta_2 = \gamma, \\ \alpha_1 &= \alpha_2 = \alpha \equiv \gamma - 1. \end{aligned}$$

Then $\bar{x} + \xi e_i \in Q$ for any $\xi \geq 0$ and

$$\bar{x} - \beta e_1 = (0, \gamma) \notin Q, \quad \bar{x} - \beta e_2 = (\gamma, 0) \notin Q,$$

$$\bar{x} - \alpha(e_1 + e_2) = (\gamma - \alpha, \gamma - \alpha) = (1, 1) \in Q.$$

Therefore, the conditions of Theorem 8.1 are satisfied and

$$\nu \geq \tfrac{\alpha_1}{\beta_1} + \tfrac{\alpha_2}{\beta_2} = 2\tfrac{\gamma - 1}{\gamma}.$$

This proves the statement since γ can be arbitrary large.
$\square$

Geometric Programming

Problem formulation:

$$\min_{x \in R^n} \; q_0(x) = \sum_{j=1}^{m_0} \alpha_{0,j} \prod_{j=1}^{n} (x^{(j)})^{\sigma_{0,j}^{(j)}}$$

$$\text{s.t} \;\; q_i(x) = \sum_{j=1}^{m_i} \alpha_{i,j} \prod_{j=1}^{n} (x^{(j)})^{\sigma_{i,j}^{(j)}} \leq 1, \tag{8.12}$$

$$i = 1 \ldots m,$$

$$x^{(j)} > 0, \quad j = 1, \ldots, n,$$

where $\alpha_{i,j}$ are some positive coefficients.

Denote $a_{i,j} = (\sigma_{i,j}^{(1)}, \ldots, \sigma_{i,j}^{(n)}) \in R^n$. Let us change the variables: $x^{(i)} = e^{y^{(i)}}$.
Then (8.12) is equivalent to a *convex* problem.

$$\min_{y \in R^n} \; \sum_{j=1}^{m_0} \alpha_{0,j} \exp(\langle a_{0,j}, y \rangle)$$

$$\text{s.t} \;\; \sum_{j=1}^{m_i} \alpha_{i,j} \exp(\langle a_{i,j}, y \rangle) \leq 1, \tag{8.13}$$

$$i = 1 \ldots m.$$

The complexity of this problem is

$$O\left(\left[\sum_{i=0}^{m} m_i \right]^{1/2} \cdot \ln \frac{1}{\epsilon} \right).$$

Approximation in L_p norms

$$\begin{aligned} & \min_x \sum_{i=1}^{m} \mid \langle a_i, x\rangle - b^{(i)} \mid^p \\ \text{s.t } & \alpha \le x \le \beta, \\ & x \in R^n, \end{aligned} \tag{8.14}$$

where $p \ge 1$.

This problem is equivalent to the following:

$$\begin{aligned} & \min_{x,y,\tau} \ \tau^{(0)}, \\ \text{s.t } & y^{(i)} = \langle a_i, x\rangle - b^{(i)}, \ i = 1, \ldots, m, \\ & \mid y^{(i)} \mid^p \le \tau^{(i)}, \ i = 1, \ldots, m, \\ & \sum_{i=1}^{m} \tau^{(i)} \le \tau^{(0)}, \\ & \alpha \le x \le \beta, \\ & x \in R^n, \ y \in R^m, \ \tau \in R^{m+1}. \end{aligned} \tag{8.15}$$

The complexity of this problem is

$$O\left(\sqrt{m+n} \cdot \ln \frac{1}{\epsilon}\right).$$

Remarks

- We have considered only the *pure* problem formulations.

  However, we can apply Interior-Point Methods to the *mixed* problems.

- We have obtained the estimates for the number of iterations of a path-following scheme.

  The real arithmetical complexity depends also on the complexity of each iteration.

  The main source of the complexity is the solution of the linear Newton system.

Choice of the minimization scheme

Problem: Approximation in L_p-norm.

$$\begin{aligned} & \min_x \sum_{i=1}^{m} \mid \langle a_i, x \rangle - b^{(i)} \mid^p \\ \text{s.t } & \alpha \le x \le \beta, \\ & x \in R^n, \end{aligned} \tag{8.16}$$

where $p \geq 1$.

Methods available:

- Ellipsoid method.
- Interior-point path-following scheme.

What scheme we should choose?

Ellipsoid Method

Number of iterations:

$$O\left(n^2 \ln \frac{1}{\epsilon}\right).$$

Complexity of the oracle:

$$O(mn)$$

arithmetic operations.

Complexity of the iteration:

$$O(n^2)$$

operations.

Total complexity:

$$O\left(n^3(m+n) \ln \frac{1}{\epsilon}\right)$$

operations.

Path-following scheme

Mediator:

$$
\begin{aligned}
& \min_{x,\tau,\xi} \ \xi, \\
\text{s.t} \ & | \langle a_i, x\rangle - b^{(i)} |^p \leq \tau^{(i)}, \ i = 1,\ldots,m, \\
& \sum_{i=1}^{m} \tau^{(i)} \leq \xi, \\
& \alpha \leq x \leq \beta, \\
& x \in R^n, \ \tau \in R^m, \ \xi \in R^1, \\
F(x,\tau) = \ & \sum_{i=1}^{m} f(\tau^{(i)}, \langle a_i, x\rangle - b^{(i)}) \\
& - \sum_{i=1}^{n} [\ln(x^{(i)} - \alpha^{(i)}) + \ln(\beta^{(i)} - x^{(i)})] \\
& - \ln(\xi - \sum_{i=1}^{m} \tau^{(i)}),
\end{aligned}
\tag{8.17}
$$

where $f(y,t) = -2\ln t - \ln(t^{2/p} - y^2)$.

Parameter: $\nu = 4m + n + 1$.

Number of iterations: $O\left(\sqrt{4m+n+1}\ln\frac{1}{\epsilon}\right)$.

Gradient: let $g_1(y,t) = f'_y(y,t)$, $g_2(y,t) = f'_t(y,t)$. Then

$$F'_x(x,\tau,\xi) = \sum_{i=1}^{m} g_1(\tau^{(i)}, \langle a_i, x\rangle - b^{(i)}) a_i$$

$$- \sum_{i=1}^{n} \left[\frac{1}{x^{(i)}-\alpha^{(i)}} - \frac{1}{\beta^{(i)}-x^{(i)}}\right] e_i,$$

$$F'_{\tau^{(i)}}(x,\tau,\xi) = g_2(\tau^{(i)}, \langle a_i, x\rangle - b^{(i)}) + \frac{1}{\xi - \sum_{i=1}^{m}\tau^{(i)}},$$

$$F'_\xi(x,\tau,\xi) = -\frac{1}{\xi - \sum_{i=1}^{m}\tau^{(i)}}.$$

Hessian: let $h_{11}(y,t) = f''_{yy}(y,t)$, $h_{12}(y,t) = f''_{yt}(y,t)$ and $h_{22}(y,t) = f''_{tt}(y,t)$. Then

$$F''_{xx}(x,\tau,\xi) = \sum_{i=1}^{m} h_{11}(\tau^{(i)}, \langle a_i, x\rangle - b^{(i)}) a_i a_i^T$$

$$+\mathrm{diag}\left[\frac{1}{(x^{(i)}-\alpha^{(i)})^2} + \frac{1}{(\beta^{(i)}-x^{(i)})^2}\right],$$

$$F''_{\tau^{(i)}x}(x,\tau,\xi) = h_{12}(\tau^{(i)}, \langle a_i, x\rangle - b^{(i)}) a_i,$$

$$F''_{\tau^{(i)},\tau^{(i)}}(x,\tau,\xi) = h_{22}(\tau^{(i)}, \langle a_i, x\rangle - b^{(i)}) + \frac{1}{(\xi - \sum_{i=1}^{m}\tau^{(i)})^2},$$

$$F''_{x,\xi}(x,\tau,\xi) = 0, \quad F''_{\tau^{(i)},\xi}(x,\tau,\xi) = -\frac{1}{(\xi - \sum_{i=1}^{m}\tau^{(i)})^2},$$

$$F''_{\xi,\xi}(x,\tau,\xi) = \frac{1}{(\xi - \sum_{i=1}^{m}\tau^{(i)})^2}.$$

Complexity of the oracle: $O(mn^2)$ a.o.

Complexity of the iteration: Solve the Newton system.

Denote: $s_i = \langle a_i, x\rangle - b^{(i)}$, $\kappa = \frac{1}{(\xi - \sum\limits_{i=1}^{m} \tau^{(i)})^2}$,

$$\Lambda_0 = \mathrm{diag}\left[\frac{1}{(x^{(i)}-\alpha^{(i)})^2} + \frac{1}{(\beta^{(i)}-x^{(i)})^2}\right]$$

$$\Lambda_1 = \mathrm{diag}\,(h_{11}(\tau^{(i)}, s_i)), \quad \Lambda_2 = \mathrm{diag}\,(h_{12}(\tau^{(i)}, s_i)),$$

$$D = \mathrm{diag}\,(h_{22}(\tau^{(i)}, s_i)), \quad i = 1, \ldots, m.$$

Let $A = (a_1, \ldots, a_m)$ and $e = (1, \ldots, 1) \in R^m$.
Then the Newton system has the following form:

$$[A(\Lambda_0 + \Lambda_1)A^T]\Delta x + A\Lambda_2 \Delta\tau = F'_x,$$

$$\Lambda_2 A^T \Delta x + [D + \kappa I_m]\Delta\tau + \kappa e \Delta\xi = F'_\tau,$$

$$\kappa\langle e, \Delta\tau\rangle + \kappa\Delta\xi = F'_\xi.$$

Therefore

$$\begin{aligned}\Delta\tau =\ & [D + \kappa I_m]^{-1}(F'_\tau - \Lambda_2 A^T \Delta x - \kappa e \Delta\xi),\\ \Delta x =\ & [A(\Lambda_0 + \Lambda_1 - \Lambda_2^2[D + \kappa I_m]^{-1})A^T]^{-1} \times \\ & \{F'_x - A\Lambda_2[D + \kappa I_m]^{-1}(F'_\tau - \kappa e \Delta\xi)\}.\end{aligned}$$

Using that, we can find $\Delta\xi$. Thus, the Newton system can be solved in $O(n^3 + mn^2)$ operations.

Total complexity of p.-f. scheme:

$$O\left(n^2(m+n)\cdot\sqrt{m+n}\cdot\ln\frac{1}{\epsilon}\right)$$

operations.

Ellipsoid method:

$$O\left(n^3(m+n)\ln\frac{1}{\epsilon}\right)$$

Conclusion:
Interior point methods are better if m is not too large:

$$m \le O(n^2).$$