Stochastics and an Optimization in Stable Dynamic Model.

– Criticism of the classical model

– Stable Dynamic Model

– Stohastic variant. Logit-model

– Potential functions

– Computational aspects

Static network assignment model (Beckmann at al,1956):

Given a network, congestion laws and an origin-destination $(\mathcal{O} - \mathcal{D})$ matrix.

Find a user-equilibrium (UE) regime.

- At UE each driver selects the shortest route (the first Wardrop principle (1952));
- the cost function is nondecreasing in the flow on the arc.

Criticism:

- a large flow corresponds to a fast movement. Then the travel time cannot be too big.

$$flow = speed \times density$$

- Triangle network

  $w_1,\ w_2$ - origin, δ - destination nodes.

$$t_1^e = \bar{t}_1; \qquad t_2^e = \bar{t}_2;$$

$$C^S = d_1\bar{t}_1 + d_2\bar{t}_2.$$

  After connection of w_1, w_2, $\bar{t}_2 > \bar{t}_1 + \bar{t}_3$:

$$t_1^e = \bar{t}_2 - \bar{t}_3;$$

$$C_1^S = d_1(\bar{t}_2 - \bar{t}_3) + d_2\bar{t}_2 > C^S.$$

Stable Dynamic model(Nesterov and de-Palma,2000):

- Given an established arc travel time pattern $t = \{t_\beta\}_{\beta \in \mathcal{A}}$ in the network $\mathcal{N}$, $t \geq \bar{t}$, each driver selects the shortest route;
- the flow f_β never exceeds $\bar{f}_\beta$.
  If $f_\beta < \bar{f}_\beta \Longrightarrow t_\beta = \bar{t}_\beta$.
  If $f_\beta = \bar{f}_\beta \Longrightarrow t_\beta \geq \bar{t}_\beta$

Let

$$T_{(i,j)}(t) = \min_{a \in \mathcal{R}_{(i,j)}} < a, t >,$$

$$C(t) = \sum_{(i,j) \in \mathcal{OD}} d_{(i,j)} T_{(i,j)}(t).$$

Theorem 1 *The arc travel time t^* and the arc flow vector f^* is an equilibrium solution of the model iif t^* is a solution to the problem:*

$$\max_t [C(t) - < \bar{f}, t - \bar{t} >: \ t \geq \bar{t}], \tag{1}$$

and $f^* = \bar{f} - s^*$, where $s^* \geq 0$ is a vector of optimal dual multipliers for the inequalities constrains in (1).

Note: At UE number of cars involved in free traffic is maximal.

Stochastic route choice model

$$r \in \mathcal{R} \qquad \{c_r\}_{r=1}^{M}$$

$$\text{travel cost function} \quad c_r(t) = \sum_{\alpha \in r} t_{(\alpha)}$$

Logit Model

$$p_r(t) = Pr(c_r(t) + \epsilon_r = \min_{q \in \mathcal{R}} (c_q(t) + \epsilon_q)).$$

$$p_r(t) = e^{-c_r(t)/\mu} / \sum_{q \in \mathcal{R}} e^{-c_q(t)/\mu}, \qquad r \in \mathcal{R}.$$

The expected arc flow vector of drivers on the route r: $f_r = f_r(t) \in R^m$:

$$f_r(t) = de^{-c_r(t)/\mu} / \sum_{q \in \mathcal{R}} e^{-c_q(t)/\mu}, \qquad r \in \mathcal{R}.$$

The expected arc low vector $f(t) \in R^m$:

$$f(t) = \sum_{r \in \mathcal{R}} f_r(t) a_r$$

The potential function

$$\psi_{\mathcal{R}}(t) = \ln \left(\sum_{r \in \mathcal{R}} e^{-c_r(t)} \right).$$

Lemma 1 *For any $\mu > 0$ and $t \in R^m$ such that $t/\mu \in int(dom \psi_{\mathcal{R}})$ we have*

$$f(t) = -d \nabla \psi_{\mathcal{R}}(t/\mu).$$

Proof.

$$c_r(t) = < a_r, t > \Longrightarrow \psi_{\mathcal{R}}(t) = \ln(\sum_{r \in \mathcal{R}} e^{-<a_r, t>}).$$

Stochastic traffic assignement

- The expected arc flow of these drivers is as follows:

$$f(t) = -d\nabla\psi_{\mathcal{R}}(t/\mu).$$

- $$\begin{array}{ll} \text{If} \quad f_\alpha < \bar{f}_\alpha \Longrightarrow tt_\alpha = \bar{t}t_\alpha; \\ \text{If} \quad f_\alpha = \bar{f}_\alpha \Longrightarrow tt_\alpha \geq \bar{t}t_\alpha \end{array} \qquad (2)$$

For each pair $(p, k) \in \mathcal{OD}$ we fix the demand flow $d_{p,k}$ and some set of routes $\mathcal{R}_{(p,k)} \in \mathcal{N}$, which connect p and k. Consider the problem:

$$\min[< \bar{f}, t > +\mu\psi(t/\mu) : \ t \geq \bar{t}], \qquad (3)$$

where $\mu > 0$ and

$$\psi(t) = \sum_{(p,k)\in\mathcal{OD}} d_{p,k} \cdot \psi_{\mathcal{R}_{p,k}}(t).$$

Theorem 2 *1. Let the demand flow be implementable by some flows $f_{p,k}$ such that*

$$\sum_{(p,k)\in OD} f_{p,k} < \bar{f}.$$

Then the problem (3) is solvable.

2. Let t^* be a solution (3). Then

$$f^*_{p,k} = -d_{p,k} \cdot \nabla\psi_{\mathcal{R}_{p,k}}(t^*/\mu), \quad (p,k) \in \mathcal{OD}.$$

These flows satisfy the corresponding demand.

3. The equilibrium arc flow $f^* = \Sigma_{(p,k)\in\mathcal{OD}} f^*_{p,k}$ satisfy the arc flow bounds. Moreover, the pair (t^*, f^*) satisfy (2).

Proof.

1. We can bound the objective function in (3) from below by some strictly increasing linear function

$$\langle \bar{f}, t\rangle + \mu\psi(t/\mu) = \langle \bar{f} - \hat{f}, t\rangle + \langle \hat{f}, t\rangle + \mu\psi(t/\mu) \geq$$

$$\geq \langle \bar{f} - \hat{f}, t\rangle + \mu\gamma \Longrightarrow \text{level sets are bounded.}$$

2. Follows from Lemma 1.

3. The solution t^* of (3)satisfies KKT conditions:

$$\bar{f} + \nabla\psi(t^*/\mu) = s^*,$$

$$s^*_\alpha \cdot (t^*_\alpha - t_\alpha) = 0,$$

where $s^* \geq 0$. Thus, if $f^*_\alpha < \bar{f}_\alpha$, we always have $s^*_\alpha > 0$ and therefore $t^*_\alpha = \bar{t}_\alpha$.

Cummulative sets of routes

$$\hat{\mathcal{R}}^L_{p,k} = \cup_{l=1}^L \mathcal{R}^l_{p,k}.$$

Let us fix some node p.

Assume we want to compute the potential functions for the cummulative sets of routes $\hat{\mathcal{R}}^L_{p,k}$, $k = 1, .., n$.

Let us fix some $\mu > 0$. Denote

$$\left.\begin{array}{l} a^l_k(t) = \mu \psi_{\mathcal{R}^l_{p,k}}(t/\mu) \\ b^l_k(t) = \mu \psi_{\hat{\mathcal{R}}^l_{p,k}}(t/\mu) \end{array}\right\} k = 1, .., n, l = 1, .., L$$

These functions can be computed by a simple recursion:

$$a^1_k(t) = b^1_k(t) = \left\{ \begin{array}{rl} -t^{\alpha[p,k]} & \text{if} \quad \alpha[p,k] \neq \emptyset, \\ -\infty, & \text{if} \quad \alpha[p,k] = \emptyset. \end{array} \right. \quad k = 1, .., n.$$

And for $l = 1, .., L-1$ we have:

$$\left.\begin{array}{l} a^{l+1}_k(t) = \mu \ln(\Sigma_{i \in I(k)} e^{(a^l_i(t) - t_{\alpha[i,k]})/\mu}) \\ b^{l+1}_k(t) = \mu \ln(e^{b^l_k(t)/\mu} + e^{a^{l+1}_k(t)/\mu}) \end{array}\right\} k = 1, .., n, \quad (4)$$

where $I(k) = \{i : \alpha[i,k] \neq \emptyset\}$.

Each step l takes $O(m)$ a.e. Thus, the computation of the values of all functions $b^L_k(t)$, $k = 1, .., n$ needs $O(Lm)$.