Космологические модели с нелокальными скалярными полями

Сергей Ю. Вернов
НИИ Ядерной Физики МГУ

доклад основан на следующих статьях

I.Ya. Aref'eva, L.V. Joukovskaya, S.V.,
J. Phys A **41** (2008) 304003, arXiv:0711.1364
S.V., Class. Quant. Grav. **27** (2010) 035006, arXiv:0907.0468
S.V., arXiv:1005.0372, S.V., arXiv:1005.5007

To specify different types of cosmic fluids one uses a relation between the pressure p and the energy density ϱ

$$p = w\varrho, \qquad p = E_k - V, \qquad \varrho = E_k + V$$

where w is the state parameter.

The spatially flat Friedmann–Robertson–Walker metric:

$$ds^2 = \, - dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{1}$$

where $a(t)$ is the scale factor, the Hubble parameter $H \equiv \dot{a}/a$.

$$w(t) = \, - 1 - \frac{2}{3}\frac{\dot{H}}{H^2} = \, - 1 + \frac{2E_k}{\varrho}. \tag{2}$$

Contemporary experiments give strong support that

$w > 0$ — Atoms. (4%)

$w = 0$ — the Cold Dark Matter. (23%)

$w < 0$ — the Dark Energy. (73%)

$$w_{DE} = -1 \pm 0.2. \tag{3}$$

We consider the case $w_{DE} < -1$. Null energy condition (NEC) is violated and there are problems of instability.

Papers about cosmological models with nonlocal fields:

I.Ya. Aref'eva, Nonlocal String Tachyon as a Model for Cosmological Dark Energy, astro-ph/0410443, 2004.
I.Ya. Aref'eva and L.V. Joukovskaya, 2005;
I.Ya. Aref'eva and A.S. Koshelev, 2006; I.Ya. Aref'eva and A.S. Koshelev, 2008;
I.Ya. Aref'eva and I.V. Volovich, 2006; I.Ya. Aref'eva and I.V. Volovich, 2007;
I.Ya. Aref'eva, 2007; A.S. Koshelev, 2007;
L.V. Joukovskaya, 2007, L.V. Joukovskaya, 2008; J.E. Lidsey, 2007;

G. Calcagni, 2006; G. Calcagni, M. Montobbio and G. Nardelli, 2007; G. Calcagni and G. Nardelli, 2007; 2009; 2010

N. Barnaby, T. Biswas and J.M. Cline, 2006; N. Barnaby and J.M. Cline, 2007; N. Barnaby and N. Kamran, 2007, 2008; N. Barnaby, 2008.

D.J. Mulryne, N.J. Nunes, 2008;

A.S. Koshelev, S.Yu. Vernov, 2009

The SFT inspired nonlocal cosmological models

From the Witten action of bosonic cubic string field theory, considering only tachyon scalar field $\phi(x)$ one obtains:

$$S=\frac{1}{g_o^2}\int d^{26}x\left[\frac{\alpha'}{2}\phi(x)\Box\phi(x)+\frac{1}{2}\phi^2(x)-\frac{1}{3}\gamma^3\Phi^3(x)-\tilde{\Lambda}\right], \qquad (4)$$

where ϕ is a scalar field,

$$\Phi=e^{k\Box}\phi, \quad k=\alpha'\ln(\gamma), \quad \gamma=\frac{4}{3\sqrt{3}}. \qquad (5)$$

g_o is the open string coupling constant, α' is the string length squared and $\tilde{\Lambda}=\frac{1}{6}\gamma^{-6}$ is added to the potential to set the local minimum of the potential to zero. The action (4) leads to equation of motion

$$(\alpha'\Box+1)e^{-2k\Box}\Phi=\gamma^3\Phi^2. \qquad (6)$$

In the majority of the SFT inspired nonlocal gravitation models the action is introduced by hand as a sum of the SFT action of tachyon field and gravity part of the action:

$$S=\frac{1}{g_o^2}\int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}R+\frac{1}{2}\phi\Box_g\phi+\frac{1}{2}\phi^2-\frac{1}{3}\gamma^3\Phi^3-\Lambda\right), \qquad (7)$$

Action (7) includes a nonlocal potential. Using a suitable redefinition of the fields, one can made the potential local, at that the kinetic term becomes nonlocal.

This nonstandard kinetic term leads to a nonlocal field behavior similar to the behavior of a phantom field, and it can be approximated with a phantom kinetic term.

The behavior of an open string tachyon can be effectively simulated by a scalar field with a phantom kinetic term.

Another type of the SFT inspired models includes nonlocal modification of gravity.

Recently G. Calcagni and G. Nardelli have considered nonlocal gravity with nonlocal scalar field (arXiv: 1004.5144).

Nonlocal action in the general form

We consider a general class of gravitational models with a nonlocal scalar field, which are described by the following action:

$$S = \int d^4x\sqrt{-g}\alpha'\left(\frac{R}{16\pi G_N} + \frac{1}{g_o^2}\left(\frac{1}{2}\phi\mathcal{F}(\Box_g)\phi - V(\phi)\right) - \Lambda\right), \qquad (8)$$

G_N is the Newtonian constant: $8\pi G_N = 1/M_P^2$,
M_P is the Planck mass.
We use the signature $(-,+,+,+)$,
$g_{\mu\nu}$ is the metric tensor,
R is the scalar curvature,
Λ is the cosmological constant.
Hereafter the d'Alembertian $\Box_g$ is applied to scalar functions and can be written as follows

$$\Box_g = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\,. \qquad (9)$$

The function $\mathcal{F}(\Box_g)$ is assumed to be an analytic function:

$$\mathcal{F}(\Box_g) = \sum_{n=0}^{\infty} f_n \Box_g^{\,n}. \tag{10}$$

Note that the term $\phi\mathcal{F}(\Box_g)\phi$ include not only terms with derivatives, but also $f_0\phi^2$.

In an arbitrary metric the energy-momentum tensor

$$T_{\mu\nu} = \, -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{g_o^2}\Big(E_{\mu\nu} + E_{\nu\mu} - g_{\mu\nu}\left(g^{\rho\sigma}E_{\rho\sigma} + W\right)\Big), \tag{11}$$

$$E_{\mu\nu} \equiv \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_\mu \Box_g^l \phi \partial_\nu \Box_g^{n-1-l}\phi, \tag{12}$$

$$W \equiv \frac{1}{2}\sum_{n=2}^{\infty} f_n \sum_{l=1}^{n-1} \Box_g^l \phi \Box_g^{n-l}\phi - \frac{f_0}{2}\phi^2 + V(\phi). \tag{13}$$

From action (8) we obtain the following equations

$$G_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} - \Lambda g_{\mu\nu}\right), \tag{14}$$

$$\mathcal{F}(\Box_g)\phi = \frac{dV}{d\phi}, \tag{15}$$

where $G_{\mu\nu}$ is the Einstein tensor.

It is a system of nonlocal nonlinear equations !!!

HOW CAN WE FIND A SOLUTION?

An algorithm of localization in the case of an arbitrary quadratic potential $V(\phi) = C_2\phi^2 + C_1\phi + C_0$.

We can change values of f_0 and Λ such that the potential takes the form $V(\phi) = C_1\phi$. So, we put $C_2 = 0$ and $C_0 = 0$.

There exist 3 cases:

- $C_1 = 0$
- $C_1 \neq 0$ **and** $f_0 \neq 0$
- $C_1 \neq 0$ **and** $f_0 = 0$

Let us start with the case $C_1 = 0$ and the equation

$$\mathcal{F}(\Box_g)\phi = 0. \tag{16}$$

We seek a particular solution of (15) in the following form

$$\phi_0 = \sum_{i=1}^{N_1} \phi_i, \tag{17}$$

$$(\Box_g - J_i)\phi_i = 0, \tag{18}$$

J_i **are roots of the characteristic equation** $\mathcal{F}(J) = 0$.

Energy–momentum tensor for special solutions

If we have *one simple root* ϕ_1 such that $\Box_g\phi_1 = J_1\phi_1$, then

$$E_{\mu\nu}(\phi_1) = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} J_1^{n-1}\partial_\mu\phi_1\partial_\nu\phi_1 = \frac{\mathcal{F}'(J_1)}{2}\partial_\mu\phi_1\partial_\nu\phi_1.$$

$$V(\phi_1) = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} J_1^{n}\phi_1^2 = \frac{J_1}{2}\sum_{n=1}^{\infty} f_n n J_1^{n-1}\phi_1^2 = \frac{J_1\mathcal{F}'(J_1)}{2}\phi_1^2.$$

In the case of *two simple roots* ϕ_1 and ϕ_2 we have

$$E_{\mu\nu}(\phi_1+\phi_2) = E_{\mu\nu}(\phi_1) + E_{\mu\nu}(\phi_2) + E_{\mu\nu}^{cr}(\phi_1,\phi_2), \tag{19}$$

where the cross term

$$E_{\mu\nu}^{cr}(\phi_1,\phi_2) = A_1\partial_\mu\phi_1\partial_\nu\phi_2 + A_2\partial_\mu\phi_2\partial_\nu\phi_1. \tag{20}$$

$$A_1 = \frac{1}{2}\sum_{n=1}^{\infty} f_n J_1^{n-1}\sum_{l=0}^{n-1}\left(\frac{J_2}{J_1}\right)^l = \frac{\mathcal{F}(J_1)-\mathcal{F}(J_2)}{2(J_1-J_2)} = 0, \tag{21}$$

$$A_2 = 0. \tag{22}$$

So, the cross term $E^{cr}_{\mu\nu}(\phi_1, \phi_2) = 0$ and

$$E_{\mu\nu}(\phi_1 + \phi_2) = E_{\mu\nu}(\phi_1) + E_{\mu\nu}(\phi_2) \tag{23}$$

Similar calculations shows

$$V(\phi_1 + \phi_2) = V(\phi_1) + V(\phi_2). \tag{24}$$

In the case of N ***simple roots*** the following formula has been obtained:

$$T_{\mu\nu} = \sum_{k=1}^{N} \mathcal{F}'(J_k) \left(\partial_\mu \phi_k \partial_\nu \phi_k - \frac{1}{2} g_{\mu\nu} \left(g^{\rho\sigma} \partial_\rho \phi_k \partial_\sigma \phi_k + J_k \phi_k^2 \right) \right). \tag{25}$$

Note that the last formula is exactly the energy-momentum tensor of many free massive scalar fields. If $\mathcal{F}(J)$ has simple real roots, then positive and negative values of $\mathcal{F}'(J_i)$ alternate, so we can obtain phantom fields.

Let $\tilde{J}_1$ is a double root. The fourth order differential equation $(\Box-\tilde{J}_1)^2\tilde{\phi}_1=0$ is equivalent to the following system of equations:

$$(\Box-\tilde{J}_1)\tilde{\phi}_1=\varphi_1, \qquad (\Box-\tilde{J}_1)\varphi_1=0. \tag{26}$$

It is convenient to write $\Box^l\tilde{\phi}_1$ in terms of the $\tilde{\phi}_1$ and φ_1:

$$\Box^l\tilde{\phi}_1=\tilde{J}_1^l\tilde{\phi}_1+l\tilde{J}_1^{l-1}\varphi_1. \tag{27}$$

Using (27) we obtain

$$E_{\mu\nu}(\tilde{\phi}_1)=B_1\partial_\mu\tilde{\phi}_1\partial_\nu\tilde{\phi}_1+B_2\partial_\mu\tilde{\phi}_1\partial_\nu\varphi_1+B_3\partial_\mu\phi_1\partial_\nu\tilde{\varphi}_1+B_4\partial_\mu\varphi_1\partial_\nu\varphi_1, \tag{28}$$

where

$$B_1=\frac{1}{2}\sum_{n=1}^{\infty}f_n\sum_{l=0}^{n-1}\tilde{J}_1^{n-1}=\frac{1}{2}\sum_{n=1}^{\infty}f_n n\tilde{J}_1^{n-1}=\frac{\mathcal{F}'(\tilde{J}_1)}{2}=0,$$

$$B_2=B_3=\frac{1}{2}\sum_{n=1}^{\infty}f_n\sum_{l=0}^{n-1}\tilde{J}_1^{n-2}(n-l-1)=\frac{\mathcal{F}''(\tilde{J}_1)}{4},$$

$$B_4=\frac{1}{2}\sum_{n=1}^{\infty}f_n\sum_{l=0}^{n-1}\tilde{J}_1^{n-3}(n-l-1)l=\frac{1}{12}\sum_{n=1}^{\infty}f_n n(n-1)(n-2)\tilde{J}_1^{n-3}=\frac{\mathcal{F}'''(\tilde{J}_1)}{12}.$$

Thus, for one double root we obtain the following result:

$$E_{\mu\nu}(\tilde{\phi}_1) = \frac{\mathcal{F}''(\tilde{J}_1)}{4}(\partial_\mu\tilde{\phi}_1\partial_\nu\varphi_1 + \partial_\mu\phi_1\partial_\nu\tilde{\varphi}_1) + \frac{\mathcal{F}'''(\tilde{J}_1)}{12}\partial_\mu\varphi_1\partial_\nu\varphi_1.$$

Similar calculations gives

$$V(\tilde{\phi}_1) = \frac{\tilde{J}_1\mathcal{F}''(\tilde{J}_1)}{2}\tilde{\phi}_1\varphi_1 + \left(\frac{\tilde{J}_1\mathcal{F}'''(\tilde{J}_1)}{12} + \frac{\mathcal{F}''(\tilde{J}_1)}{4}\right)\varphi_1^2. \qquad (29)$$

For one single root and one double root we obtain:

$$E_{\mu\nu}(\tilde{\phi}_1 + \phi_2) = E_{\mu\nu}(\tilde{\phi}_1) + E_{\mu\nu}(\phi_2) + E^{cr}_{\mu\nu}(\tilde{\phi}_1, \phi_2), \qquad (30)$$

where

$$E^{cr}_{\mu\nu}(\tilde{\phi}_1, \phi_2) = B_5\partial_\mu\tilde{\phi}_1\partial_\nu\phi_2 + B_6\partial_\nu\tilde{\phi}_1\partial_\mu\phi_2 + B_7\partial_\mu\varphi_1\partial_\nu\phi_2 + B_8\partial_\nu\varphi_1\partial_\mu\phi_2.$$

It is easy to calculate:

$$E^{cr}_{\mu\nu}(\tilde{\phi}_1, \phi_2) = 0. \qquad (31)$$

Using similar calculations we obtain that

$$V(\tilde{\phi}_1 + \phi_2) = V(\tilde{\phi}_1) + V(\phi_2). \qquad (32)$$

For any analytical function $\mathcal{F}(J)$, which has simple roots J_i and double roots $\tilde{J}_k$, the energy–momentum tensor

$$T_{\mu\nu}\left(\phi_0\right) = T_{\mu\nu}\left(\sum_{i=1}^{N_1}\phi_i + \sum_{k=1}^{N_2}\tilde{\phi}_k\right) = \sum_{i=1}^{N_1} T_{\mu\nu}(\phi_i) + \sum_{k=1}^{N_2} T_{\mu\nu}(\tilde{\phi}_k), \qquad (33)$$

where

$$T_{\mu\nu} = \frac{1}{g_o^2}\Big(E_{\mu\nu} + E_{\nu\mu} - g_{\mu\nu}\left(g^{\rho\sigma}E_{\rho\sigma} + W\right)\Big), \qquad (34)$$

$$E_{\mu\nu}(\phi_i) = \frac{\mathcal{F}'(J_i)}{2}\partial_\mu\phi_i\partial_\nu\phi_i, \quad W(\phi_i) = \frac{J_i\mathcal{F}'(J_i)}{2}\phi_i^2, \quad \mathcal{F}' \equiv \frac{d\mathcal{F}}{dJ} \qquad (35)$$

$$E_{\mu\nu}(\tilde{\phi}_k) = \frac{\mathcal{F}''(\tilde{J}_k)}{4}\Big(\partial_\mu\tilde{\phi}_k\partial_\nu\varphi_k + \partial_\nu\tilde{\phi}_k\partial_\mu\varphi_k\Big) + \frac{\mathcal{F}'''(\tilde{J}_k)}{12}\partial_\mu\varphi_k\partial_\nu\varphi_k, \qquad (36)$$

$$W(\tilde{\phi}_k) = \frac{\tilde{J}_k\mathcal{F}''(\tilde{J}_k)}{2}\tilde{\phi}_k\varphi_k + \left(\frac{\tilde{J}_k\mathcal{F}'''(\tilde{J}_k)}{12} + \frac{\mathcal{F}''(\tilde{J}_k)}{4}\right)\varphi_k^2. \qquad (37)$$

Consider the following local action

$$S_{loc} = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G_N} - \Lambda\right) + \sum_{i=1}^{N_1} S_i + \sum_{k=1}^{N_2} \tilde{S}_k, \tag{38}$$

where

$$S_i = -\frac{1}{g_o^2}\int d^4x\sqrt{-g}\frac{\mathcal{F}'(J_i)}{2}\left(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i + J_i\phi_i^2\right),$$

$$\tilde{S}_k = -\frac{1}{g_o^2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\left(\frac{\mathcal{F}''(\tilde{J}_k)}{4}\left(\partial_\mu\tilde{\phi}_k\partial_\nu\varphi_k + \partial_\nu\tilde{\phi}_k\partial_\mu\varphi_k\right) + \right.\right.$$

$$\left.\left. + \frac{\mathcal{F}'''(\tilde{J}_k)}{12}\partial_\mu\varphi_k\partial_\nu\varphi_k\right) + \frac{\tilde{J}_k\mathcal{F}''(\tilde{J}_k)}{2}\tilde{\phi}_k\varphi_k + \left(\frac{\tilde{J}_k\mathcal{F}'''(\tilde{J}_k)}{12} + \frac{\mathcal{F}''(\tilde{J}_k)}{4}\right)\varphi_k^2\right).$$

Remark 1. If $\mathcal{F}(J)$ has an infinity number of roots then one nonlocal model corresponds to infinity number of different local models. In this case the initial nonlocal action (8) generates infinity number of local actions (38).

Remark 2. We should prove that the way of localization is self-consistent. To construct local action (38) we assume that equations (18) are satisfied. Therefore, the method of localization is correct only if these equations can be obtained from the local action S_{loc}. The straightforward calculations show that

$$\frac{\delta S_{loc}}{\delta \phi_i} = 0 \;\Leftrightarrow\; \Box_g \phi_i = J_i \phi_i; \qquad \frac{\delta S_{loc}}{\delta \tilde{\phi}_k} = 0 \;\Leftrightarrow\; \Box_g \varphi_k = \tilde{J}_k \varphi_k. \qquad (39)$$

$$\frac{\delta S_{loc}}{\delta \varphi_k} = 0 \qquad \Leftrightarrow \qquad \Box_g \tilde{\phi}_k = \tilde{J}_k \tilde{\phi}_k + \varphi_k. \qquad (40)$$

We obtain from S_{loc} the Einstein equations as well:

$$G_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu}(\phi_0) - \Lambda g_{\mu\nu}\right), \qquad (41)$$

where ϕ_0 is given by (17) and $T_{\mu\nu}(\phi_0)$ can be calculated by (33).

Any solutions of system (39)–(41) are particular solutions of the initial nonlocal system (14)–(15).

Let us consider the model with action (8) in the case $C_1 \neq 0$.

If $f_0 \neq 0$, then we introduce a new scalar field

$$\chi = \phi - \frac{C_1}{f_0} \tag{42}$$

and get the energy–momentum tensor in the form (34) with

$$E_{\mu\nu} = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_\mu \Box_g^l \chi \partial_\nu \Box_g^{n-1-l} \chi, \tag{43}$$

$$W = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=1}^{n-1} \Box_g^l \chi \Box_g^{n-l} \chi - \frac{f_0}{2}\chi^2 + \frac{C_1^2}{2f_0}. \tag{44}$$

It is easy to see that

$$\mathcal{F}(\Box_g)\phi = C_1 \qquad \Longleftrightarrow \qquad \mathcal{F}(\Box_g)\chi = 0. \tag{45}$$

If $f_0 = 0$, then $J = 0$ is a root of the characteristic equation $\mathcal{F}(J) = 0$. It is easy to show, that the function

$$\tilde{\chi} = \phi_0 + \psi, \tag{46}$$

where ϕ_0 and ψ are solutions of the following equations

$$\mathcal{F}(\Box_g)\phi_0 = 0, \qquad \Box_g\psi = \frac{C_1}{f_1}. \tag{47}$$

is a solution for

$$\mathcal{F}(\Box_g)\tilde{\chi} = C_1. \tag{48}$$

In the case $f_1 \neq 0$ the order of the root $J = 0$ is equal to one. The function ϕ_0 is given by (17), but the sum do not include ϕ_{i_0}, which corresponds to the root $J = 0$, because this function can not be separated from ψ.

It is easy to show:

$$T_{\mu\nu}(\tilde{\chi}) = T_{\mu\nu}(\psi) + T_{\mu\nu}(\phi_0), \tag{49}$$

$$W(\psi) = C_1\psi + \frac{f_2C_1^2}{2f_1^2}, \quad E_{\mu\nu}(\psi) = \frac{1}{2}f_1\partial_\mu\psi\partial_\nu\psi, \tag{50}$$

For an arbitrary quadratic potential $V(\phi) = C_2\phi^2 + C_1\phi + C_0$ ***there exists the following algorithm of localization***:

- **Change values of** f_0 **and** Λ **such that the potential takes the form** $V(\phi) = C_1\phi$**.**
- **Find roots of the function** $\mathcal{F}(J)$ **and calculate orders of them.**
- **Select an finite number of simple roots.**
- **Construct the corresponding local action. In the case** $C_1 = 0$ **one should use formula (38). In the case** $C_1 \neq 0$ **and** $f_0 \neq 0$ **one should use (38) with the replacement of the scalar field** ϕ **by** χ**. In the case** $C_1 \neq 0$ **and** $f_0 = 0$ **the local action is the sum of (38) and (in the case of simple root** $J = 0$**)**

$$S_\psi = -\frac{1}{2g_o^2}\int d^4x\sqrt{-g}\left(f_1 g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + 2C_1\psi + \frac{f_2C_1^2}{f_1^2}\right).$$

- **Vary the obtained local action and get a system of the Einstein equations and equations of motion.**
- **Seek solutions of the obtained local system.**

Exact Solutions in the FRW metric

Let us consider the Friedmann equations, which corresponds to a real simple root J_1:

$$\begin{cases} 3H^2 = \dfrac{4\pi G\mathcal{F}'(J_1)}{g_o^2}\left(\dot{\phi}^2 + J_1\phi^2\right) + 8\pi G\Lambda, \\ \dot{H} = -\dfrac{4\pi G\mathcal{F}'(J_1)}{g_o^2}\dot{\phi}^2, \end{cases} \tag{51}$$

a dot denotes a time derivative. Exact real solutions of this system are as follows:

At $J_1 > 0$

$$\phi(t) = \pm\frac{\sqrt{3J_1}g_o^2}{6\pi G\mathcal{F}'(J_1)}(t - t_0), \qquad H(t) = -\frac{J_1 g_o^2}{6\pi G\mathcal{F}'(J_1)}(t - t_0), \tag{52}$$

where t_0 is an arbitrary constant. These solutions exist only at

$$\Lambda = -\frac{J_1 g_o^2}{24G^2\pi^2\mathcal{F}'(J_1)}\,. \tag{53}$$

At $J_1 = 0$ the type of solution depends on sign of Λ:

- **In the case $\Lambda < 0$**

$$H_2(t) = -\frac{2\sqrt{-6\pi G\Lambda}}{3}\tan\left(2\sqrt{-6\pi G\Lambda}(t-t_0)\right), \tag{54}$$

$$\phi_2(t) = C_1 \pm \sqrt{\frac{g_o^2}{12\pi G\mathcal{F}'(0)}}\text{arctanh}\left(\sin\left(2\sqrt{-6\pi G\Lambda}(t-t_0)\right)\right). \tag{55}$$

- $\Lambda = 0$

$$H(t) = -\frac{1}{3(t-t_0)}, \quad \phi(t) = C_1 \pm \frac{\sqrt{3}g_o}{\sqrt{\pi G\mathcal{F}'(0)}}\ln(t-t_0). \tag{56}$$

- **If $\Lambda > 0$, then we obtain solutions:**

$$H_1(t) = \frac{2\sqrt{6\pi G\Lambda}}{3}\tanh\left(2\sqrt{6\pi G\Lambda}\,(t-t_0)\right), \qquad (57)$$

$$\phi_1(t) = C_1 \pm \sqrt{\frac{-\,g_o^2}{12\pi G\mathcal{F}'(0)}}\arctan\left(\sinh\left(2\sqrt{6\pi G\Lambda}\,(t-t_0)\right)\right) \qquad (58)$$

Using $\tanh(t+i\pi/2)=\coth(t)$**, one gets a new real solution.**

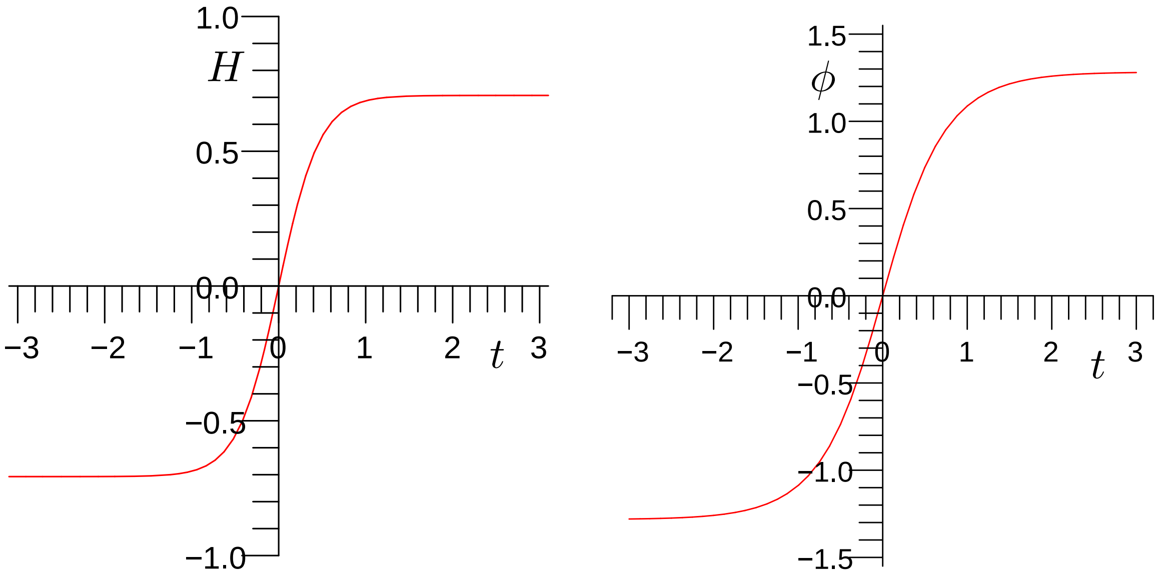

Рис. 1: The functions $H_1(t)$ (right) and $\phi_1(t)$ (left) at $t_0=0$ and $C_2=0$.

SOLUTIONS FOR EQUATIONS OF MOTION

(S.V. arXiv:1005.5007)

Let us consider nonlocal Klein–Gordon equation in the case of an arbitrary potential:

$$\mathcal{F}(\Box_g)\phi = V'(\phi), \tag{59}$$

where prime is a derivative with respect to ϕ. A particular solution of (59) is a solution of the following system:

$$\sum_{n=0}^{N-1} f_n \Box_g^n \phi = V'(\phi) - C, \qquad f_N \Box_g^N \phi = C, \tag{60}$$

where $N-1$ is a natural number, C is an arbitrary constant.

In the case $f_1 \neq 0$ we can choose $N = 2$.

In the spatially flat FRW metric with the interval:

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{61}$$

where $a(t)$ **is the scale factor, we obtain from (60):**

$$f_1\Box_g\phi = -f_1\left(\ddot{\phi} + 3H\dot{\phi}\right) = V'(\phi) - f_0\phi - C, \qquad f_2\Box_g^2\phi = C. \tag{62}$$

The Hubble parameter

$$H = -\frac{1}{3\dot{\phi}}\left(\ddot{\phi} + \tilde{V}'(\phi) - \frac{C}{f_1}\right), \tag{63}$$

where

$$\tilde{V}'(\phi) \equiv \frac{1}{f_1}\left(V'(\phi) - f_0\phi\right). \tag{64}$$

Equation

$$(\partial_t^2 + 3H\partial_t)\left(\ddot{\phi} + 3H\dot{\phi}\right) = \frac{C}{f_2}, \tag{65}$$

is as follows

$$(\partial_t^2 + 3H\partial_t)\tilde{V}' = \tilde{V}'''\dot{\phi}^2 + \tilde{V}''(\ddot{\phi} + 3H\dot{\phi}) = -\frac{C}{f_2}. \tag{66}$$

We eliminate H and obtain

$$\dot{\phi}^2 = \frac{1}{\tilde{V}'''}\left(\tilde{V}''\tilde{V}' - \frac{C}{f_1}\tilde{V}'' - \frac{C}{f_2}\right). \tag{67}$$

The obtained equation can be solved in quadratures. Its general solution depend on two arbitrary parameters C and t_0, which corresponds to the time shift.

It allows to find solutions for an arbitrary potential $V(\phi)$, with the exception of linear and quadratic potentials.

Note that we do not consider other Einstein equations. In distinguish to the localization method, which allows to localize all Einstein equations, this method solves only the field equation, whereas the obtained solutions maybe do not solve other equations.

Our results may be regarded as examples of exact solutions of nonlocal nonlinear equations on a curved background.

Perhaps, the adding of other type of matter can give an exact solution of the system of all Einstein equations.

CUBIC POTENTIAL

The case of cubic potential is is connected with the bosonic string field theory. Let us find solutions (59) for

$$V(\phi) = B_3\phi^3 + B_2\phi^2 + B_1\phi + B_0, \tag{68}$$

where B_0, B_1, B_2, and B_3 are arbitrary constants, but $B_3 \neq 0$. For this potential we get (67) in the following form

$$\dot{\phi}^2 = 4C_3\phi^3 + 6C_2\phi^2 + 4C_1\phi + C_0, \tag{69}$$

where

$$C_0 = \frac{(B_1 - C)(2B_2 - f_0)}{6f_1B_3} - \frac{Cf_1^2}{6f_1f_2B_3}, \qquad C_2 = \frac{2B_2 - f_0}{4f_1}, \tag{70}$$

$$C_1 = \frac{6B_3(B_1 - C) + (2B_2 - f_0)^2}{24f_1B_3}, \qquad C_3 = \frac{3B_3}{4f_1}. \tag{71}$$

Note, that $C_3 \neq 0$ since $B_3 \neq 0$. Using the transformation

$$\phi = \frac{1}{2C_3}(2\xi - C_2), \tag{72}$$

we get the following equation

$$\dot{\xi}^2 = 4\xi^3 - g_2\xi - g_3, \tag{73}$$

where

$$g_2 = \frac{(2B_2 - f_0)^2 - 12B_3(B_1 - C)}{16f_1^2}, \quad g_3 = 2C_1C_2C_3 - C_2^3 - C_0C_3^2 = -\frac{3B_3C}{32f_2f_1}.$$

A solution of equation (73) is either the Weierstrass elliptic function

$$\xi(t) = \wp(t - t_0, g_2, g_3), \tag{74}$$

or a degenerate elliptic function.

Let us consider degenerated cases. At $g_2 = 0$ and $g_3 = 0$

$$\phi_1 = \frac{1}{C_3(t-t_0)^2} - \frac{C_2}{2C_3} = \frac{4f_1}{3B_3(t-t_0)^2} - \frac{2B_2 - f_0}{6B_3}. \tag{75}$$

$$H_1 = \frac{5}{3(t-t_0)}. \tag{76}$$

We are of interest to obtain a bounded solution, which tends to a finite limit at $t \to \infty$. We have obtained such solutions in the following form

$$\phi_2 = D_2 \tanh(\beta(t-t_0))^2 + D_0, \tag{77}$$

$$D_2 = \frac{4}{3B_3} f_1\beta^2, \qquad D_0 = \frac{1}{18B_3}\left(3(f_0 - 2B_2) - 16 f_1\beta^2\right), \tag{78}$$

where β is a root of the following equation

$$1024 f_2 f_1 \beta^6 + 576 f_1^2 \beta^4 + 324 B_3 B_1 - 27(2B_2 - f_0)^2 = 0. \tag{79}$$

Bounded real solutions for equation (69) correspond to real root of equations (79). Pure image root of (79) correspond to unbounded real solutions for equation (69), because $\tanh(\beta t)^2 = -\tan(\mathrm{i}\beta t)^2$. The solution ϕ_2 exists at

$$C = \frac{1}{36B_3}\left(64 f_1^2\beta^4 - 3(2B_2 - f_0)^2 + 36 B_3 B_1\right). \tag{80}$$

$$H_2 = \frac{\beta(2\cosh(\beta t)^2 - 3)}{3\cosh(\beta t)\sinh(\beta t)} -$$

$$- \frac{3B_3(D_2\tanh(\beta t)^2 + D_0)^2 + (2B_2 - f_0)(D_2\tanh(\beta t)^2 + D_0) + B_1}{6 f_1 D_2 \beta \tanh(\beta t)(1 - \tanh(\beta t)^2)}.$$

Conclusions

We have studied the SFT inspired nonlocal models with quadratic potentials and obtained:

Local and Nonlocal Einstein equations have one and the same solutions.

Nonlocality arises in the case of $\mathcal{F}(\Box_g)$ with an infinite number of roots.

One system of Nonlocal Einstein equations $\Leftrightarrow$ Infinity number of systems of local Einstein equations.

In the case of an arbitrary potential, but linear or quadratic potential, we have obtained a particular solution of $\mathcal{F}(\Box_g)\phi = V'(\phi)$

In the Friedmann–Robertson–Walker metric the proposed method for the search of exact solutions for field equation allows to get in quadratures solutions, which depend on two arbitrary parameters.

Exact solutions have been found for a cubic potential.